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Analysis of the Mean Value Theorem and Rolle's Theorem in Holomorphic Function

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ABSTRACT

The Mean Value Theorem and Rolle's Theorem are one of the fundamental concepts in real analytical mathematics related to the derivative value of functions. This research examines both theorems on holomorphic functions in the complex plane. Holomorphic functions are also known as analytic functions, which are complex functions of one variable that have derivatives at every point in their domain. The results of the analysis show that there are fundamental differences in the application of the two theorems to holomorphic functions which can be seen in terms of the separation real and imaginary components, analytic property of the function, and the domain requirements that must be met. This research also discusses some variations of the Mean Value Theorem in holomorphic functions, such as Flett's Theorem and Myers Theorem and it can be shown that in complex functions the Mean Value Theorem and Rolle's Theorem are mutually equivalent.

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KEYWORDS: Mean Value Theorem, Rolle's Theorem, Holomorphic Functions, Complex Function of One Variable

I. INTRODUCTION

Complex analysis is vital branch of mathematical analysis that extends the concept of calculus into the complex domain. When viewed in the field of calculus, it focuses on the study of functions with studies basic concepts, theories, properties of complex variables, encompassing their limits, continuity, differentiability, and integrability. Complex analysis is an important foundation for various field of science, ranging from engineering, physics, economics, applied mathematics, such as optimization, calculus, mathematical modelling, and other field of sciences [1].

These functions, particularly holomorphic functions, namely complex functions that have derivatives of functions that have complex-valued derivatives at every point in their domain. These functions have unique and interesting properties that are far more rigid and structured compared to their real-valued distinguish them from real functions because in the structure of complex numbers there are real and imaginary parts, which enables the definition of stronger complex differentiability [2]. Two fundamental concepts in real analysis that play an important role in studying the properties of functions include the Rolle's Theorem and the Mean Value Theorem which is a special case of the Rolle's Theorem. It is evident that both theorems possess numerous applications in the domain of classical analysis. These applications include, but are not limited to, non-linear equations and optimization problems [3]. The Mean Value Theorem (MVT) as it is known in its modern form now was first introduced by French mathematician Augustin-Louis Cauchy (1789-1857). However, the basic concept of the Theorem dates back to the 18th century. The Mean Value Theorem is widely applied in various fields, one of which is in engineering to analyze the rate of movement and calculate the average velocity [4]. Michael Rolle (1652-1719) was a French mathematician who first introduced the properties or conditions of the derivative of a polynomial function named Rolle's Theorem. Rolle's Theorem becomes a special case of the Mean Value Theorem when the change value of the function is zero. Rolle's Theorem states that for any differentiable function with equal values at the endpoints of a closed interval, there exists at least one point where the derivative vanishes. Meanwhile, the MVT generalizes this by relating the average rate of change of a function to its instantaneous rate of change at some point in the interval. When applied to complex functions, these two theorems cannot be used in their usual form due to the nature of differentiation in complex function require the fulfilment of the Cauchy-Riemann equation. It means that in the domain of complex functions that are differentiable over the entire complex plane, the concepts of value change and the point where the derivative is zero need different approaches [5].

There have been several previous studies that have discussed the Mean Value Theorem and Rolle's Theorem separately. For example, research by Huyen and Cakmak extended the Mean Value Theorem on holomorphic functions of a complex variable [5] [6] with specific conditions and introduced generalized forms such as Flett's Theorem and Myers Theorem. It also shows the results of proof and equivalence between the Mean Value Theorem and Rolle's Theorem on holomorphic functions of one complex variable [5] [6]. Evard and Jafari introduced an extension of Rolle's Theorem in holomorphic functions of a complex variable and explained how the Mean Value Theorem for holomorphic functions is also derived from Rolle's Theorem [7] While these generalizations provide insight, a comprehensive comparative analysis between Rolle's Theorem and the MVT within the framework of holomorphic functions remains limited. This study aims to fill that gap by exploring how these classical real-variable theorems adapt to the structure of complex-valued functions. By examining their conditions, graphical implications, and mathematical consequences, this paper contributes to a deeper understanding of the behavior of holomorphic functions and enriches the theoretical foundation of complex analysis.

II. METHODOLOGY

This study adoptes a qualitative analytical approach based on a literature review and comparative theoretical analysis to explore the generalization of Rolle's Theorem and the Mean Value Theorem (MVT) in the context of holomorphic functions. The methods applied in this study are designed to investigate the extent to which these real analysis theorems can be meaningfully interpreted or extended within the field of complex analysis.

A. Research Approach

The research is structured as a conceptual exploration, where known mathematical theorems in real-variable analysis are formally restated and then analyzed within the domain of complex variables. Specifically, the objective of this study is to offer a comprehensive reconstruction of the classical proofs and geometric interpretations of Rolle's Theorem and the Mean Value Theorem in real analysis. A review of the properties and differentiability criteria of holomorphic functions is conducted. Also, a study of the applicability and limitations of these theorems when extended to the complex domain is also necessary, particularly under the constraints imposed by the Cauchy-Riemann equations.

B. Literature Sources

The core materials used in this study include Textbooks on real and complex analysis (e.g., Bartle, Rudin, Churchill & Brown) that provide formal definitions, theorems, and classical proofs. The following peer-reviewed journal articles propose extensions of the Mean Value Theorem and Rolle's Theorem in the complex plane works by Cakmak and Tiryaki [5] and Huyen [6]. Supplementary visual illustrations and plots were created using *Python* (*Matplotlib*) to geometrically represent complex function mappings and derivative behaviors.

C. Scope and Limitations

The scope of the study is constrained to single-variable holomorphic functions defined over domains in the complex plane. Multivariable or non-holomorphic complex functions are not included in this study. The analysis is constrained to mathematical logic and geometric illustration, excluding empirical simulation or experimental validation.

III. RESULT

This section presents the results of a theoretical analysis that explores the extension of Rolle's Theorem and the Mean Value Theorem (MVT) from real analysis to holomorphic functions in complex analysis. The discussion places particular emphasis on structural differences, conceptual generalizations, and the conditions under which these theorems hold in complex domains.

A. Fundamental Properties of Holomorphic Functions

Definition 3.1 [8]. Suppose $f: D \to \mathbb{C}$, a function f is called holomorphic di domain D if f has a derivative at every point in D.

Theorem 3.1 [9] [10] [11]. (Necessary Conditions for Holomorphic Functions). Suppose $f: D \to \mathbb{C}$, if $f(z) = u(x, y) + iv(x, y)$ is a holomorphic function in a domain then it satisfies the Cauchy-Riemann equation i.e:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof:

Suppose $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$. Suppose $z_0 = x_0 + iy_0 \in D$, then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the approach path $z \to z_0$ is taken horizontally (when $x \to 0$, $y = 0$), then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

If the approach path $z \to z_0$ is taken vertically (when $y \to 0$, $x = 0$), then

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

In order for a function f to have a complex (*holomorphic*) derivative, the partial derivatives with respect to x and y from different approximation paths must be the same (singular/unique). Consequently, the Cauchy-Riemann Theorem must be

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

B. Rolle's Theorem on Complex Functions

Definition 3.2 [5] [6]. (Rolle's and MVT Theorem in Complex Plane). Let a be a complex number $a, b \in \mathbb{C}$ and $a \neq b$. The closed interval $[a, b]$ in $\mathbb{C}$ is a line segment that can be expressed as a set of
$[a, b] = \{z | z = (1-\alpha)a + \alpha b\}, \alpha \in [0,1]$.

Theorem 3.2 [5] [6]. Suppose f be a holomorphic function defined on an convex subset $D \subset \mathbb{C}$. If $a, b \in D$ and $a \neq b$ satisfy $f(a) = f(b) = 0$, then there exist $z_1, z_2 \in D$ such that $Re(f'(z_1)) = 0$ and $Im(f'(z_2)) = 0$.

Proof:

Define an auxiliary function $g(z)$ that satisfies the conditions of Rolle's Theorem where $g(a) = g(b) = 0$. Let

$$g(z) = f(z).\overline{b-a}$$

and $z = (1-\alpha)a + \alpha b$, $\alpha \in [0,1]$. Let $h(\alpha)$ is a function with real variables defined by:

$$h(\alpha) = g\big((1-\alpha)a + \alpha b\big) = f\big((1-\alpha)a + \alpha b\big)\big(\overline{b-a}\big)$$

then

$$h(\alpha) = g\big(z(\alpha)\big) = f\big(z(\alpha)\big)\big(\overline{b-a}\big).$$

Given that $f(z)$ is a holomorphic function on D then $g(z)$ is also holomorphic on D. Because $f(a) = f(b) = 0$ and $\alpha \in [0,1]$, it has

$$h(0) = g(a) = f(a) \cdot \overline{b-a} = 0$$

it follow that if $f(a) = 0$, then $h(0) = 0$, and

$$h(1) = g(b) = f(b) \cdot \overline{b-a} = 0$$

so that if $f(b) = 0$, then $h(1) = 0$. Thus, $h(0) = h(1) = 0$, so the function $h(\alpha)$ satisfies the condition of Rolle's Theorem for real functions) [12]. It means that $h(\alpha) = 0$, ($h(\alpha)$ is zero) at both ends of the interval $[a, b]$, then there is a point z, so $h'(\alpha) = 0$.

$$h(\alpha) = Re\Big(g\big(z(\alpha)\big)\Big) + Im\Big(g\big(z(\alpha)\big)\Big)$$

$$h(\alpha) = Re\Big(f(1-\alpha)a + \alpha b \cdot \big(\overline{b-a}\big)\Big) + Im\Big(f(1-\alpha)a + \alpha b \cdot \big(\overline{b-a}\big)\Big)$$

Let

$$u(\alpha) = Re\Big(f\big((1-\alpha)a + \alpha b\big)\big(\overline{b-a}\big)\Big)$$

and

$$v(\alpha) = Im\Big(f\big((1-\alpha)a + \alpha b\big)\big(\overline{b-a}\big)\Big)$$

thus

$$h(\alpha) = u(\alpha) + iv(\alpha).$$

obtained that

$u(0) = Re\Big(f(a)\big(\overline{b-a}\big)\Big) = 0$ dan $u(1) = Re\Big(f(b)\big(\overline{b-a}\big)\Big) = 0$,

$v(0) = Im\Big(f(a)\big(\overline{b-a}\big)\Big) = 0$ dan $v(1) = Im\Big(f(b)\big(\overline{b-a}\big)\Big) = 0$,

with $0 \leq \alpha \leq 1$, so we have

$$h'(\alpha) = g'\big((1-\alpha)a + \alpha b\big)$$
$$= f'\big((1-\alpha)a + \alpha b\big)\big(\overline{b-a}\big)(b-a)$$
$$h'(\alpha) = f'\big((1-\alpha)a + \alpha b\big)\big(\overline{b-a}\big)(b-a)$$

Given that $(b-a)\big(\overline{b-a}\big)$ is equal to $(b-a)^2$, so

$$h'(\alpha) = f'\big((1-\alpha)a + \alpha b\big) \cdot |b-a|^2$$
$$h'(\alpha) = f'\big(z(\alpha)\big) \cdot |b-a|^2.$$

Let

$$u'(\alpha) = Re\Big(f'\big(z(\alpha)\big)\Big)$$
$$u'(\alpha) = Re\Big(f'\big((1-\alpha)a + \alpha b\big)\Big)$$

and

$$v'(\alpha) = Im\Big(f'\big(z(\alpha)\big)\Big)$$
$$v'(\alpha) = Im\Big(f'\big((1-\alpha)a + \alpha b\big)\Big)$$

with $\alpha \in [0,1]$, then

$$h'(\alpha) = \big(u'(\alpha) + iv'(\alpha)\big)|b-a|^2$$

$h'(\alpha) = u'(\alpha)|b-a|^2 + iv'(\alpha)|b-a|^2$.

We have

$u(0) = Re(f'(a)|b-a|^2) = 0$ dan $u(1) = Re(f'(b)|b-a|^2) = 0$,

$v(0) = Im(f'(a)|b-a|^2) = 0$ dan $v(1) = Im(f'(b)|b-a|^2) = 0$.

$u(0) = u(1) = 0$,

$v(0) = v(1) = 0$.

By using Rolle's Theorem for real functions [12] for $u(\alpha)$ on $[0,1]$, there exists $\beta \in [0,1]$ such that

$$u'(\beta) \cdot |b-a|^2 = 0$$

there exists $z_1 = (1-\beta)a + \beta b \in (a, b)$, such that $u'(\beta) = 0$ or can be expressed as

$u'(\beta) = Re\big(f'\big((1-\beta)a + \beta b\big) \cdot |b-a|^2\big) = Re(f'(z_1) \cdot |b-a|^2) = 0$.

$\exists z_1 = (1-\beta)a + \beta b \in [a, b]$, so

$$Re\big(g'(z_1)\big) = u'(\beta)|b-a|^2 = 0$$

$Re\big(g'(z_1)\big) = Re\big(f'(z_1)\big)\, Re\big(\overline{b-a}\big) = 0$,

then

$Re\big(f'(z_1)\big) = 0$.

By using Rolle's Theorem for real functions for $v(\alpha)$ on $[0,1]$, there exists $\gamma \in [0,1]$ such that

$$v'(\gamma) \cdot |b-a|^2 = 0$$

there exists $z_2 = (1-\gamma)a + \gamma b \in (a, b)$, such that $v'(\gamma) = 0$ or can be expressed as

$v'(\gamma) = Im\big(f'\big((1-\gamma)a + \gamma b\big) \cdot |b-a|^2\big) = Im(f'(z_2) \cdot |b-a|^2) = 0$.

$\exists z_2 = (1-\gamma)a + \gamma b \in [a, b]$, so

$$Im\big(g'(z_2)\big) = v'(\gamma)|b-a|^2 = 0$$

$Im\big(g'(z_2)\big) = Im\big(f'(z_2)\big)\, Im\big(\overline{b-a}\big) = 0$,

then

$Im\big(f'(z_2)\big) = 0$.

Rolle's Theorem in complex plane can be proven by proving the two-variable Rolle's Theorem which is converted into a one-variable Rolle's Theorem. The following is proved if the functions $u(x,y)$ dan $v(x,y)$ are expressed as functions of one variable in $u(x)$.
$(x,y) \in [(1-\alpha)a_1 + \alpha a_2, (1-\alpha)b_1 + \alpha b_2]$, $0 \le \alpha \le 1$.
For $\alpha = 0$ then $z = a$ and for $\alpha = 1$ then $z = b$, $\alpha \in [0,1]$.
Let
$a = a_1 + ib_1$, $a = (a_1, b_1)$
and
$b = a_2 + ib_2$, $b = (a_2, b_2)$.

Let
$f(z) = u(x,y) + iv(x,y)$ and $f'(z) = u_x(x,y) + iv_x(x,y)$.
We have

$$f(a) = u(a_1,b_1) + iv(a_1,b_1) = 0$$

so $u(a_1,b_1) = 0$ and $v(a_1,b_1) = 0$.

$$f(b) = u(a_2,b_2) + iv(a_2,b_2) = 0$$

so $u(a_2,b_2) = 0$ and $v(a_2,b_2) = 0$. We know that,
$[a,b] = \{z | z = (1-\alpha)a + \alpha b\}$.
$z = x + iy = (1-\alpha)(a_1 + ib_1) + \alpha(a_2 + ib_2) = \big((1-\alpha)a_1 + \alpha a_2 + i(1-\alpha)b_1 + \alpha b_2\big)$,
thus

$$x = (1-\alpha)a_1 + \alpha a_2$$

and
$y = (1-\alpha)b_1 + \alpha b_2$.
We have

$$x = a_1 - \alpha a_2 + \alpha a_2 = a_1 + \alpha(a_2 - a_1),\ \ a_1 \ne a_2.$$

$$x = a_1 + \alpha(a_2 - a_1)$$

$$x - a_1 = \alpha(a_2 - a_1)$$

Because $a_2 \ne a_1$ then $a_2 - a_1 \ne 0$, so

$$\alpha = \tfrac{x-a_1}{a_2-a_1},\ \alpha \in [0,1],\ x \in (a_1,a_2).$$

We have

$$y = \left(1 - \frac{x-a_1}{a_2-a_1}\right)b_1 + \left(\frac{x-a_1}{a_2-a_1}\right)b_2$$

Because $f(z) = u(x,y) + iv(x,y)$, so function $u(x,y)$ can be expressed as a function of one variable, i.e:
$u(x,y) = u\left(x, y = \left(1 - \tfrac{x-a_1}{a_2-a_1}\right)b_1 + \left(\tfrac{x-a_1}{a_2-a_1}\right)b_2\right) = u(x)$, $x \in (a_1,a_2)$.
then
$f(z) = u(x) + iv(x)$, $x \in (a_1,a_2)$.
By using Rolle's Theorem for real functions [12] there exists $c_1 \in (a_1,a_2)$ such that $u'(c_1) = 0$. Let

$$d_1 = \left(1 - \frac{c_1-a_1}{a_2-a_1}\right)b_1 + \left(\frac{c_1-a_1}{a_2-a_1}\right)b_2$$

so $u_x(c_1,d_1) = 0$, we have

$$u_x(c_1,d_1) = 0$$

$u_x(c_1) = u_x\left(c_1, d_1 = \left(1 - \tfrac{c_1-a_1}{a_2-a_1}\right)b_1 + \left(\tfrac{c_1-a_1}{a_2-a_1}\right)b_2\right) = 0$.
c $z_1 = c_1 + id_1 \in [a,b]$ such that
$Re\big(f'(z_1)\big) = u_x(c_1,d_1) = 0$.
In the same way, $v(x,y)$ can be expressed as a function of one variable, i.e:
$v(x,y) = v\left(x, y = \left(1 - \tfrac{x-a_1}{a_2-a_1}\right)b_1 + \left(\tfrac{x-a_1}{a_2-a_1}\right)b_2\right) = v(x)$, $x \in (a_1,a_2)$.
By using Rolle's Theorem for real functions there exists $c_2 \in (a_1,a_2)$ such that $v'(c_2) = 0$. Let

$$d_2 = \left(1 - \frac{c_2-a_1}{a_2-a_1}\right)b_1 + \left(\frac{c_2-a_1}{a_2-a_1}\right)b_2$$

so $v_x(c_2,d_2) = 0$, we have

$$v_x(c_2,d_2) = 0$$

$$v_x(c_2) = v_x\left(c_2, d_2 = \left(1 - \tfrac{c_2-a_1}{a_2-a_1}\right)b_1 + \left(\tfrac{c_2-a_1}{a_2-a_1}\right)b_2\right) = 0.$$

Therefore, there exists $z_2 = c_2 + id_2 \in [a,b]$ such that
$Im\big(f'(z_2)\big) = v_x(c_2,d_2) = 0$.
Because $f(a) = f(b) = 0$, so

$$f(a) = u(a_1,b_1) + iv(a_1,b_1) = 0$$

thus $u(a_1,b_1) = v(a_1,b_1) = 0$, and

$$f(b) = u(a_2,b_2) + iv(a_2,b_2) = 0$$

thus $u(a_2,b_2) = v(a_2,b_2) = 0$.
Therefore,

$$u(a_1,b_1) = u(a_2,b_2) = 0$$

then
$u(a_1) = u(a_2) = 0$ (a function in $u(x)$).
and

$$v(a_1,b_1) = v(a_2,b_2) = 0$$

then
$v(a_1) = v(a_2) = 0$ (a function in $v(x)$).
We know that $\frac{du}{dx}$ exists on (a_1,a_2) with $a = (a_1,b_1)$ and $b = (a_2,b_2)$ where $u(a_1) = u(a_2) = 0$.
And $\frac{dv}{dx}$ exists on (a_1,a_2) with $a = (a_1,b_1)$ and $b = (a_2,b_2)$ where $v(a_1) = v(a_2) = 0$.
Because the first partial derivatives u_x, u_y, v_x, v_y exists on the interval (a,b), then $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exists on (a_1,a_2) with $u(a_1) = u(a_2) = 0$ and $v(a_1) = v(a_2) = 0$.
Therefore, by using both in real functions for $u(x)$ and $v(x)$ on (a_1,a_2), then there exists $z_1, z_2 \in [a,b]$, such that

$$u'(z_1) = \frac{u(a_2) - u(a_1)}{a_2 - a_1} = 0 \Rightarrow u'(z_1) = 0$$

and

$$v'(z_2) = \frac{v(a_2) - v(a_1)}{a_2 - a_1} = 0 \Rightarrow v'(z_2) = 0$$

Therefore, it can be concluded by Rolle's Theorem that for functions of one variable, in this case functions in $u(x)$ dan $v(x)$ there exists $z_1 = c_1 + id_1 \in [a,b]$ and $z_2 = c_2 + id_2 \in [a,b]$, i.e. $\exists z_1 = (c_1,d_1), \exists z_2 = (c_2,d_2)$, then

$$u_x(c_1,d_1) = u_x(z_1) = 0$$

and

$$v_x(c_2,d_2) = v_x(z_2) = 0$$

It is proven that

$$f'(z) = u_x + iv_x$$

$$f'(z) = u_x(c_1,d_1) + iv_x(c_2,d_2)$$

$$f'(z) = u_x(z_1) + iv_x(z_2)$$

$f'(z) = 0$.

It can also be concluded that Rolle’s Theorem is proven in functions of one variable as follows.

$\exists c_1, d_1 \in (a_1, a_2)$, thus $Re(f'(z_1)) = \frac{\partial u}{\partial x}(c_1, d_1) = u_x(c_1, d_1) = 0 = u_x(z_1)$

and

$\exists c_2, d_2 \in (a_1, a_2)$, thus $Im(f'(z_2)) = \frac{\partial v}{\partial x}(c_2, d_2) = v_x(c_2, d_2) = 0 = v_x(z_2)$

then

$$f'(z) = u_x(x, y) + iv_x(x, y) = u_x(x) + iv_x(x)$$

satisfies the Cauchy-Riemann equation. So, it can be proven that

$$0 = u_x(z_1) = Re(f'(z_1)) = 0$$

$$0 = v_x(z_2) = Im(f'(z_2)) = 0$$

For example of Rolle’s Theorem, consider the function $f(z) = z^3 - 3z$, let be $a = -\sqrt{3}$ and $b = \sqrt{3}$ or consider the function $f(z) = z(z - (3 + 3i))$ with $a = 0 + 0i$ and $b = 3 + 3i$.

C. Mean Value Theorem (MVT) on Complex Functions

Theorem 3.3 [5] [6]**.** Suppose f be a holomorphic function defined on an convex subset $D \subset \mathbb{C}$. If $a, b \in D$ and $a \neq b$ satisfy $f(a) = f(b) = 0$, then there exist $z_1, z_2 \in (a, b)$ on the complex line segment between a and b such that

$$Re(f'(z_1)) = Re\left(\frac{f(b) - f(a)}{b - a}\right)$$

denotes the real part of the derivative of the function at point z_1, i.e. $f'(z_1)$ is equal to the mean change in the real part of the function, and

$$Im(f'(z_2)) = Im\left(\frac{f(b) - f(a)}{b - a}\right)$$

denotes the real part of the derivative of the function at point z_2, i.e. $f'(z_2)$ is equal to the mean change of the imaginary part of the function.

Proof:

Define the auxiliary function $g(z)$ to eliminate the linear component in $f(z)$ on the interval $[a, b]$ as follows:

$$g(z) = f(z) - f(a) - \frac{f(b) - f(a)}{b - a}(z - a)$$

Because $f(z)$ is a holomorphic function in the domain D and $g(z)$ is a linear combination of $f(z)$ and $\frac{f(b)-f(a)}{b-a}(z - a)$, so $g(z)$ is also holomorphic in D. Consider the following property of $g(z)$:

Substitution $z = a$

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

so that obtained

$g(a) = 0$.

Substitution $z = b$ results in the cancellation of all components in the expression $g(z)$.

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a)$$
$$= f(b) - f(a) - (f(b) - f(a))$$

so that obtained

$g(b) = 0$.

So, we have $g(a) = g(b) = 0$, and according to Theorem 3.3 (Rolle's Theorem on the complex plane) $g(z)$ is a holomorphic function that is zero at both ends of the interval. According to the properties of holomorphic functions, the Mean Value Theorem can be applied to find z_1, z_2 in the segment (a, b) where $Re(g'(z_1)) = 0$ and $Im(g'(z_2)) = 0$. Since the auxiliary function $g(z)$ is a holomorphic function, then $g'(z)$ is also holomorphic, then based on Rolle's Theorem for real functions there exists a point $z_1 \in (a, b)$ such that

$Re(g'(z_1)) = 0$.

From the definition of $g(z)$, it can be obtained that the derivative of the function $g(z)$ is

$g'(z) = f'(z) - \frac{f(b)-f(a)}{b-a}$.

By using Rolle’s Theorem for real function [12], ensure that there is a point $z_1 \in (a, b)$, such that if $Re(g'(z_1)) = 0$, then

$$Re(g'(z_1)) = Re\left(f'(z_1) - \frac{f(b) - f(a)}{b - a}\right) = 0$$

$$Re\left(f'(z_1) - \frac{f(b) - f(a)}{b - a}\right) = 0$$

$$Re(f'(z_1)) - Re\left(\frac{f(b) - f(a)}{b - a}\right) = 0$$

$$Re(f'(z_1)) = Re\left(\frac{f(b) - f(a)}{b - a}\right)$$

Furthermore, for the imaginary part of the function $g(z)$ can be obtained according Rolle's Theorem for real functions [12] which ensures that there exists a point $z_2 \in (a, b)$ such that

$$Im(g'(z_2)) = 0$$

so that if the imaginary part $Im(g'(z_2)) = 0$, then

$$Im(g'(z_2)) = Im\left(f'(z_2) - \frac{f(b) - f(a)}{b - a}\right) = 0$$

$$Im\left(f'(z_2) - \frac{f(b) - f(a)}{b - a}\right) = 0$$

$$Im(f'(z_2)) - Im\left(\frac{f(b) - f(a)}{b - a}\right) = 0$$

$Im(f'(z_2)) = Im\left(\frac{f(b)-f(a)}{b-a}\right)$.

Geometrically, this theorem states that along the path of a line segment between two points a and b in the complex plane, there exists z_1 where the local rate of change (real part) is parallel to the instantaneous gradient and z_2 where the local rate of change (imaginary part) is parallel to the instantaneous gradient. The mapping of a holomorphic function f is a transformation of a line segment (a, b) into a curve in the complex plane.

For example of Mean Value Theorem, consider the function $f(z) = z^2 + iz$ is defined on (all) complex numbers $D = \mathbb{C}$. Suppose two points $a = 1 + i$ and $b = 2 + 2i$ on the line segment (a, b). If f is holomorphic over the entire domain $\mathbb{C}$ or $D = \mathbb{C}$, then f is called an *entire function* [13].

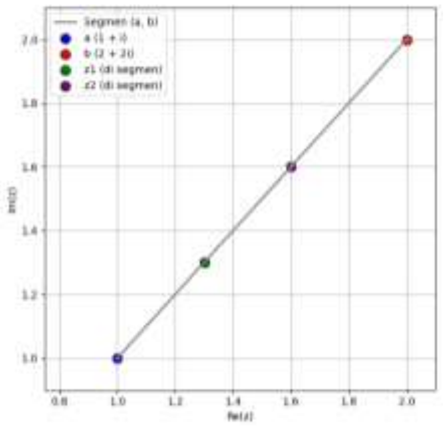

Figure 3.1 Geometric illustration of points $z_1, z_2 \in (a, b)$ in the complex plane

D. Flett’s Theorem for Complex Functions

In the domain of complex analysis, research is conducted on complex numbers of the form $z = x + iy$, where x denotes the real part and y represents the imaginary part. The imaginary unit, denoted by i satisfies the equation $i^2 = -1$ [14].

Theorem 3.4 (Davitt, Powers, Riedel, and Sahoo's Theorem) [5] [15]. Suppose f is a holomorphic function defined on an open convex set $D_f \subseteq \mathbb{C}$. if $a, b \in D_f$ and $a \neq b$, then there exist two points $z_1, z_2 \in (a, b)$ such that

$$Re\left(f'(z_1)\right) = \frac{\langle b-a, f(z_1) - f(a)\rangle}{\langle b-a, z_1 - a\rangle} + \frac{1}{2}\frac{Re\left(f'(b) - f'(a)\right)}{b-a}(z_1 - a)$$

and

$$Im\left(f'(z_2)\right) = \frac{\langle b-a, -i[f(z_2) - f(a)]\rangle}{\langle b-a, z_2 - a\rangle} + \frac{1}{2}\frac{Im\left(f'(b) - f'(a)\right)}{b-a}(z_2 - a)$$

Proof:

Suppose the holomorphic function $f(z)$ can be expressed as follows:

$$f(z) = U(x,y) + iV(x,y)$$

where $U(x,y) = Re\left(f(z)\right)$ is the real part of $f(z)$ and $V(x,y) = Im\left(f(z)\right)$ is the imaginary part of $f(z)$.

Suppose $z = a + t(b-a)$, so $f(z)$ can be rewritten as

$f\left(a + t(b-a)\right) = U\left(a + t(b-a)\right) + iV\left(a + t(b-a)\right)$.

Define the auxiliary function $\Phi: [0,1] \to \mathbb{R}$ as follows:

$$\Phi(t) = \left\langle b-a, f\left(a + t(b-a)\right)\right\rangle$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in complex space. Based on the definition of inner product, the auxiliary function $\Phi(t)$ can be written as follows.

$$\Phi(t) = Re\left((b-a)\cdot\overline{f\left(a + t(b-a)\right)}\right)$$

Since $f\left(a + t(b-a)\right) = U\left(a + t(b-a)\right) + iV\left(a + t(b-a)\right)$, the complex conjugate is

$$\overline{f\left(a + t(b-a)\right)} = U\left(a + t(b-a)\right) - iV\left(a + t(b-a)\right),$$

so that

$$\Phi(t) = Re\left((b-a)\cdot\left(U\left(a + t(b-a)\right) - iV\left(a + t(b-a)\right)\right)\right)$$

where $(b-a)(U - iV) = (b-a)U - i(b-a)V$, then by taking the real part, the real form of the function $\Phi(t)$ is obtained

$$\Phi(t) = Re\left((b-a)U\left(a + t(b-a)\right)\right) + Im\left((b-a)V\left(a + t(b-a)\right)\right)$$

Thus, the auxiliary function $\Phi(t)$ is a real function on the interval $[0,1]$ and can be expressed in its explicit form as follows.

$\Phi(t) = Re\left[(b-a)U\left(a + t(b-a)\right)\right] + Im\left[(b-a)V\left(a + t(b-a)\right)\right]$.

Because the function f has a derivative on the convex set D_f Consequently $f(z)$ has a derivative on $[a,b] \subseteq D_f$ for every $a, b \in D_f$. Therefore, $\Phi(t)$ has a derivative for every $t \in [0,1]$. By using the Cauchy-Riemann equation, we obtain

$$\Phi'(t) = \left\langle b-a, (b-a)f'\left(a + t(b-a)\right)\right\rangle$$

$$\Phi'(t) = Re\left((b-a)\overline{(b-a)f'\left(a + t(b-a)\right)}\right)$$

Given that $\langle b-a, b-a\rangle = (b-a)\overline{(b-a)} = |b-a|^2$, so that

$$Re\left(|b-a|^2\overline{f'\left(a + t(b-a)\right)}\right).$$

Recall that the derivative of $f(z) = U(x,y) + iV(x,y)$ is $f'(z) = U_x + iV_x$. Suppose $z = a + t(b-a)$, then

$$Re\left(|b-a|^2\left(\frac{\partial U(z)}{\partial x} + i\frac{\partial V(z)}{\partial x}\right)\right) = Re\left(|b-a|^2\left(\frac{\partial U(z)}{\partial x} - i\frac{\partial V(z)}{\partial x}\right)\right) = |b-a|^2\left(\frac{\partial U(z)}{\partial x}\right) = |b-a|^2\cdot U_x$$

so that

$$\Phi'(t) = |b-a|^2\, Re\left(f'(z)\right).$$

Then, Theorem 3.4 is proved by applying auxiliary function $\Phi(t)$ which is a real-valued function. By applying for $\Phi(t)$ at $t \in [0,1]$, there exists $t_1 \in (0,1)$ such that it holds that

$$\Phi(t_1) - \Phi(0) = \Phi'(t_1)(t_1 - 0) - \frac{1}{2}\frac{\Phi'(1) - f'(0)}{1-0}(t_1 - 0)^2$$

$$\Phi'(t_1)(t_1) = \Phi(t_1) - \Phi(0) + \frac{1}{2}[\Phi'(1) - \Phi'(0)](t_1)^2$$

for a given $t_1 \in (0,1)$. By substituting $\Phi'(t_1) = |b-a|^2\, Re\left(f'(z_1)\right)$, one obtains

$$(t_1)|b-a|^2 Re\big(f'(z_1)\big) = \Phi(t_1)-\Phi(0) + \frac{1}{2}[\Phi'(1)-\Phi'(0)](t_1)^2$$

where $z_1 = a + t_1(b-a)$ and $t_1 \in [0,1]$.

It can be noted that,

$$(t_1)|b-a|^2 = (t_1)(b-a)\overline{(b-a)}$$
$$(t_1)|b-a|^2 = (b-a)\cdot\overline{(t_1)(b-a)}$$

Because $z_1 = a + t_1(b-a)$, then

$$(t_1)(b-a) = bt_1 - at_1 == a + t_1(b-a) - a = z_1 - a,$$

so that

$$(t_1)|b-a|^2 = (b-a)\overline{z_1 - a}$$
$$(t_1)|b-a|^2 = \langle b-a, z_1-a\rangle.$$

It was found that,

$$Re\big(f'(z_1)\big) = \frac{\Phi(t_1)-\Phi(0)}{(t_1)|b-a|^2} + \frac{1}{2}\frac{\Phi'(1)-\Phi'(0)}{(t_1)|b-a|^2}(t_1)^2$$
$$Re\big(f'(z_1)\big) = \frac{\Phi(t_1)-\Phi(0)}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{\Phi'(1)-\Phi'(0)}{|b-a|^2}(t_1)$$

Based on the auxiliary function $\Phi(t)$ at $t \in [0,1]$, that:

$$\Phi(t) = \big\langle b-a, f\big(a+t(b-a)\big)\big\rangle$$

then

$$\Phi(t_1) = \big\langle b-a, f\big(a+t_1(b-a)\big)\big\rangle = \langle b-a, f(z_1)\rangle$$

and

$$\Phi(0) = \big\langle b-a, f\big(a+0(b-a)\big)\big\rangle = \langle b-a, f(a)\rangle.$$

Further,

$$\Phi'(t_1) = |b-a|^2 Re\big(f'(z_1)\big)$$

so

$$\Phi'(1) = |b-a|^2 Re\big(f'(b)\big)$$

and

$$\Phi'(0) = |b-a|^2 Re\big(f'(a)\big).$$

Therefore, it can be obtained

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(z_1)\rangle - \langle b-a, f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{|b-a|^2 Re\big(f'(b)\big) - |b-a|^2 Re\big(f'(a)\big)}{|b-a|^2}(t_1)$$
$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(z_1)\rangle - \langle b-a, f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{Re\big(f'(b)\big) - Re\big(f'(a)\big)}{b-a}(t_1)(b-a)$$
$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{Re\big(f'(b)\big)-\big(f'(a)\big)}{b-a}(z_1-a).$$

In the same way, it can be proved for the imaginary part, namely $Im\big(f'(z_2)\big)$. Suppose $g(z) = -if(z)$ and

$f(z) = U(x,y) + iV(x,y)$ with $Re\big(f(z)\big) = U(x,y)$ and $Im\big(f(z)\big) = V(x,y)$, then it is known that

$$g(z) = -i\big(U(x,y) + iV(x,y)\big)$$
$$g(z) = -iU(x,y) + V(x,y)$$

where $Re\big(g(z)\big) = V(x,y)$ and $Im\big(g(z)\big) = -U(x,y)$, so that we get

$$Re\big(-if(z)\big) = Im\big(f(z)\big).$$

By applying the same result as in the real part to the function $g(z)$, a point z_2 can be found such that the following applies

$$Re\big(g'(z_2)\big) = \frac{\langle b-a, g(z_2)-g(a)\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{Re\big(f'(b)\big)-\big(f'(a)\big)}{b-a}(z_2-a)$$

where $z_2 = a + t_2(b-a)$ and $t_2 \in [0,1]$.

Since $g(z) = -if(z)$, then $g'(z) = -if'(z)$, thus

$$Re\big(g'(z_2)\big) = Im\big(f'(z_2)\big).$$

We obtained,

$$Im\big(f'(z_2)\big) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{Im\big(f'(b)\big)-\big(f'(a)\big)}{b-a}(z_2-a).$$

Hence Theorem 3.4 is proved i.e. if the function f is holomorphic and defined on f by applying Flett's Theorem on real-valued functions.

E. Myers Theorem for Complex Functions

Theorem 3.5 [5] [6]. Suppose $f: I = [a,b] \to \mathbb{R}$, if f is a real-valued and continuous function on the closed interval $[a,b] \subset \mathbb{R}$ and is differentiable on the open interval $(a,b) \subset \mathbb{R}$, and $f'(a) = f'(b)$ then there exists a point $c \in (a,b)$ so

$$f(b) - f(c) = f'(c).(b-c)$$

Theorem 3.6 (Myers Theorem on Complex Valued Functions) [5]. Let f be a holomorphic function defined on an open convex set $D_f \subseteq \mathbb{C}$. if $a, b \in D_f$ and $a \neq b$, then there exist $z_1, z_2 \in (a,b)$ such that

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{Re\big(f'(b)-f'(a)\big)}{b-a}(b-z_1)$$

and

$$Im\big(f'(z_2)\big) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{Im\big(f'(b)-f'(a)\big)}{b-a}(b-z_2)$$

Proof:

Suppose the function f is holomorphic and defined on $D_f \subseteq \mathbb{C}$. Here it is shown that there exist two points $z_1, z_2 \in (a,b)$ such that it holds that

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{Re\big(f'(b)-f'(a)\big)}{b-a}(b-z_1)$$

and

$$Im\big(f'(z_2)\big) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{Im\big(f'(b)-f'(a)\big)}{b-a}(b-z_2)$$

Suppose the holomorphic function $f(z)$ can be expressed as follows.

$$f(z) = U(x,y) + iV(x,y)$$

where $U(x,y) = Re\big(f(z)\big)$ is the real part of $f(z)$ and $V(x,y) = Im\big(f(z)\big)$ is the imaginary part of $f(z)$.
Suppose $z = a + t(b-a)$, so $f(z)$ can be rewritten as

$$f\big(a + t(b-a)\big) = U\big(a + t(b-a)\big) + iV\big(a + t(b-a)\big).$$

Define the auxiliary function $\Phi: [0,1] \to \mathbb{R}$ as follows.

$$\Phi(t) = \big\langle b-a, f\big(a + t(b-a)\big)\big\rangle$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in complex space. Based on the definition of inner product in complex numbers, the auxiliary function $\Phi(t)$ can be written as follows.

$$\Phi(t) = Re\left((b-a)\cdot\overline{f\big(a+t(b-a)\big)}\right)$$

Because $f\big(a + t(b-a)\big) = U\big(a + t(b-a)\big) + iV\big(a + t(b-a)\big)$, the complex conjugate is

$$\overline{f\big(a+t(b-a)\big)} = U\big(a + t(b-a)\big) - iV\big(a + t(b-a)\big),$$

thus

$$\Phi(t) = Re\left((b-a)\cdot\left(U\left(a+t(b-a\right) - iV\big(a+t(b-a)\big)\right)\right)\right)$$

where $(b-a)(U-iV) = (b-a)U - i(b-a)V$, then by taking the real part, the real form of the function $\Phi(t)$ is obtained

$$\Phi(t) = Re\Big((b-a)U\big(a+t(b-a)\big)\Big) + Im\Big((b-a)V\big(a+t(b-a)\big)\Big)$$

Thus, the auxiliary function $\Phi(t)$ is a real function on the interval $[0,1]$ and can be expressed in its explicit form as follows.

$$\Phi(t) = Re\big[(b-a)U\big(a+t(b-a)\big)\big] + Im\big[(b-a)V\big(a+t(b-a)\big)\big].$$

Because the function f has a derivative on the convex set D_f then $f(z)$ has a derivative on $[a,b] \subseteq D_f$ for every $a, b \in D_f$. Therefore, $\Phi(t)$ has a derivative for every $t \in [0,1]$. By using the Cauchy-Riemann equation, we obtain

$$\Phi'(t) = \big\langle b-a, (b-a)f'\big(a+t(b-a)\big)\big\rangle$$

$$\Phi'(t) = Re\left((b-a)\overline{(b-a)f'\big(a+t(b-a)\big)}\right).$$

Given that $\langle b-a, b-a\rangle = (b-a)\overline{(b-a)} = |b-a|^2$, so that

$$Re\left(|b-a|^2\overline{f'\big(a+t(b-a)\big)}\right).$$

Recall that the derivative of $f(z) = U(x,y) + iV(x,y)$ is $f'(z) = U_x + iV_x$. suppose $z = a + t(b-a)$, then

$$Re\left(|b-a|^2\left(\overline{\frac{\partial U(z)}{\partial x} + i\frac{\partial V(z)}{\partial x}}\right)\right) = Re\left(|b-a|^2\left(\frac{\partial U(z)}{\partial x} - i\frac{\partial V(z)}{\partial x}\right)\right) = |b-a|^2\left(\frac{\partial U(z)}{\partial x}\right)$$

so that obtained

$$\Phi'(t) = |b-a|^2\, Re\big(f'(z)\big).$$

Then, Theorem 3.6 is proved by applying auxiliary function $\Phi(t)$ which is a real-valued function.
By applying $\Phi(t)$ at $t \in [0,1]$, there exists $t_1 \in (0,1)$ such that it holds that

$$\Phi(1) - \Phi(t_1) = \Phi'(t_1)(1-t_1) + \frac{1}{2}\frac{\Phi'(1) - f'(0)}{1-0}(1-t_1)^2$$

$$\Phi'(t_1)(1-t_1) = \Phi(1) - \Phi(t_1) - \frac{1}{2}[\Phi'(1) - \Phi'(0)](1-t_1)^2$$

for a given $t_1 \in (0,1)$. By substituting $\Phi'(t_1) = |b-a|^2\, Re\big(f'(z_1)\big)$, obtains

$$(1-t_1)|b-a|^2\, Re\big(f'(z_1)\big) = \Phi(1) - \Phi(t_1) - \frac{1}{2}[\Phi'(1) - \Phi'(0)](1-t_1)^2$$

where $z_1 = a + t_1(b-a)$ dan $t_1 \in [0,1]$.
It can be noted that,

$$(1-t_1)|b-a|^2 = (1-t_1)(b-a)\overline{(b-a)}$$

$$(1-t_1)|b-a|^2 = (b-a)\cdot\overline{(1-t_1)(b-a)}.$$

Since $z_1 = a + t_1(b-a)$, then

$$(1-t_1)(b-a) = b - a - bt_1 + at_1 = (b-a) - t_1(b-a) = b - \big(a + t_1(b-a)\big) = b - z_1,$$

so that obtained

$$(1-t_1)|b-a|^2 = (b-a)\overline{b-z_1}$$

$$(1-t_1)|b-z_1|^2 = \langle b-a, b-z_1\rangle.$$

It was found that,

$$Re\big(f'(z_1)\big) = \frac{\Phi(1) - \Phi(t_1)}{(1-t_1)|b-a|^2} - \frac{1}{2}\frac{\Phi'(1) - \Phi'(0)}{(1-t_1)|b-a|^2}(1-t_1)^2$$

$$Re\big(f'(z_1)\big) = \frac{\Phi(1) - \Phi(t_1)}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{\Phi'(1) - \Phi'(0)}{|b-a|^2}(1-t_1)$$

Based on the auxiliary function $\Phi(t)$ at $t \in [0,1]$, namely:

$$\Phi(t) = \big\langle b-a, f\big(a+t(b-a)\big)\big\rangle$$

then

$$\Phi(1) = \big\langle b-a, f\big(a+1(b-a)\big)\big\rangle = \langle b-a, f(b)\rangle$$

and

$$\Phi(t_1) = \big\langle b-a, f\big(a+t_1(b-a)\big)\big\rangle = \langle b-a, f(z_1)\rangle.$$

Further,

$$\Phi'(t_1) = |b-a|^2\, Re\big(f'(z_1)\big)$$

then

$$\Phi(1) = |b-a|^2\, Re\big(f'(b)\big)$$

and

$$\Phi(0) = |b-a|^2\, Re\big(f'(a)\big).$$

Therefore, it can be obtained

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(b)\rangle - \langle b-a, f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{|b-a|^2\,Re\big(f'(b)\big) - |b-a|^2\,Re\big(f'(a)\big)}{|b-a|^2}(1-t_1)$$

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(b)\rangle - \langle b-a, f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{Re\big(f'(b)\big) - Re\big(f'(a)\big)}{b-a}(1-t_1)(b-a)$$

$$Re\big(f'(z_1)\big) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{Re\big(f'(b)\big) - \big(f'(a)\big)}{b-a}(b-z_1).$$

In the same way, it can be proved for the imaginary part, namely $Im\big(f'(z_2)\big)$. Suppose $g(z) = -if(z)$ and $f(z) = U(x,y) + iV(x,y)$ with $Re\big(f(z)\big) = U(x,y)$ and $Im\big(f(z)\big) = V(x,y)$, then it is known that

$$g(z) = -i\big(U(x,y) + iV(x,y)\big)$$
$$g(z) = -iU(x,y) + V(x,y)$$

with $Re\big(g(z)\big) = V(x,y)$ and $Im\big(g(z)\big) = -U(x,y)$, thus we have

$$Re\big(-if(z)\big) = Im\big(f(z)\big).$$

By applying the same result as in the real part to the function $g(z)$, a point z_2 can be found such that the following holds

$$Re\big(g'(z_2)\big) = \frac{\langle b-a, g(b)-g(z_2)\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{Re\big(f'(b)\big) - \big(f'(a)\big)}{b-a}(b-z_2)$$

with $z_2 = a + t_2(b-a)$ and $t_2 \in [0,1]$.

Since $g(z) = -if(z)$, then $g'(z) = -if'(z)$, thus

$$Re\big(g'(z_2)\big) = Im\big(f'(z_2)\big)$$

We obtained,

$$Im\big(f'(z_2)\big) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{Im(f\prime(b)) - (f\prime(a))}{b-a}(b-z_2).$$

Hence Theorem 3.6 is proved i.e. if the function f is holomorphic and defined on $D_f \subseteq \mathbb{C}$ by applying Myers' Theorem on real-valued functions in Theorem 3.5.

F. Equivalence of Rolle and Mean Value Theorem in Holomorphic Function

The following shows the proof of the equivalence property based on the Mean Value Theorem and Rolle’s Theorem. To prove it, it is assumed that a function f satisfies all the conditions of both theorems. Define the auxiliary function $g(z)$ using the concept of determinant of a 3×3 matrix as follows.

$$g(z) := \frac{1}{a-b}\begin{vmatrix} f(z) & f(a) & f(b) \\ z & a & b \\ 1 & 1 & 1 \end{vmatrix}$$

$$g(z) = \frac{1}{a-b}\left(f(z)\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} - f(a)\begin{vmatrix} z & b \\ 1 & 1 \end{vmatrix} + f(b)\begin{vmatrix} z & a \\ 1 & 1 \end{vmatrix}\right)$$

$$g(z) = \frac{1}{a-b}[f(z)(a-b) - f(a)(z-b) + f(b)(z-a)]$$

$$g(z) = \frac{f(z)(a-b) - f(a)(z-b) + f(b)(z-a)}{a-b}$$

$g(z)$ can be expressed as a linear combination of $f(z)$, $f(a)$, and $f(b)$, thus obtained a linear combination of the values of the function f at three points z, a, and b.

$$g(z) = f(z) - f(a)\frac{z-b}{a-b} + f(b)\frac{z-a}{a-b}$$

We know that

$$g(a) = f(a) - f(a)\frac{a-b}{a-b} + f(b)\frac{a-a}{a-b}$$
$$g(a) = f(a) - f(a)$$
$$g(a) = 0$$

and

$$g(b) = f(b) - f(a)\frac{b-b}{a-b} + f(b)\frac{b-a}{a-b}$$
$$g(b) = f(b) + \big(-f(b)\big)$$

$g(b) = 0.$

So, we have $g(a) = g(b) = 0$. Let $f(a) = f(b)$, then $g'(z) = f'(z)$.

By using Rolle’s Theorem in holomorphic functions we need to showing the condition of $g(z)$ at $z = a$ and $z = b$ $(g(a) = g(b) = 0)$ similar to the condition of Rolle’s Theorem in holomorphic functions where g has the same value at both points, then there exists $z \in (a,b)$ such that $f'(z) = 0$.

$$g(z) = f(z) - f(a)\frac{z-b}{a-b} + f(b)\frac{z-a}{a-b}$$

$$g'(z) = f'(z) - \left(\frac{f(a)+f(b)}{a-b}\right) \times (-1)$$

$$g'(z) = f'(z) - \frac{f(b)-f(a)}{b-a}$$

for each $z \in D_f$. The derivative of a complex function $g(z)$ expresses the difference between the derivative of $f'(z)$ and the mean change of function f on the complex interval $[a,b]$. Because $f(a) = f(b) = 0$ based on Rolle’s Theorem in real function, we have

$$g'(z) = f'(z) - 0 = f'(z)$$

Substitute $g'(z) = 0$, to show the equivalence relation of Rolle’s Theorem with the Mean Value Theorem, then

$$0 = f'(z) - \frac{f(b)-f(a)}{b-a}$$

$$f'(z) = \frac{f(b)-f(a)}{b-a}$$

Based on Rolle’s Theorem in holomorphic functions, so that $g'(z_1) = Re\big(f'(z_1)\big) = 0$ and $g'(z_2) = Im\big(f'(z_2)\big) = 0$, then

$$g'(z_1) = f'(z_1) - \frac{f(b)-f(a)}{b-a}$$

and

$$g'(z_2) = f'(z_2) - \frac{f(b)-f(a)}{b-a}$$

then the Mean Value Theorem on holomorphic functions applies as follows.

$$0 = g'(z_1) = f'(z_1) - \frac{f(b)-f(a)}{b-a}$$

and

$$0 = g'(z_2) = f'(z_2) - \frac{f(b) - f(a)}{b - a}$$

thus

$$Re\big(f'(z_1)\big) = Re\big(g'(z_1)\big) = Re\left(\frac{f(b) - f(a)}{b - a}\right)$$

and

$$Im\big(f'(z_2)\big) = Im\big(g'(z_2)\big) = Im\left(\frac{f(b) - f(a)}{b - a}\right)$$

Therefore, based on the Rolle's Theorem and the Mean Value Theorem, the relationship between the two theorems in holomorphic functions is obtained as follows.

$$0 = Re\big(g'(z_1)\big) = Re\big(f'(z_1)\big) - Re\left(\frac{f(b) - f(a)}{b - a}\right)$$

and

$$0 = Im\big(g'(z_1)\big) = Im\big(f'(z_1)\big) - Im\left(\frac{f(b) - f(a)}{b - a}\right)$$

If the Mean Value Theorem is valid, then a point can be found where the derivative of the function is zero which implies the Rolle's Theorem is also valid. The symmetry between the Mean Value Theorem and Rolle's Theorem in the complex plane is also fulfilled, i.e. if one theorem holds, then the other also holds or if the Mean Value Theorem holds in a complex function, then Rolle's Theorem must also be valid.

IV. DISCUSSION AND CONCLUSION

The objective of this study was to examine the applicability of Rolle's Theorem and the Mean Value Theorem (MVT) within the complex analysis domain. The findings demonstrate that while both theorems hold under well-defined conditions in real analysis, their direct application in the complex domain is not always valid due to the rigidity of holomorphic differentiability, defined by the Cauchy-Riemann equations.

A. Discussion

The classical Rolle's Theorem, which ensures the existence of a point where the derivative is zero when the endpoints are equal, does not hold unconditionally for holomorphic functions. However, through separation into real and imaginary parts, a modified version is obtained: there exist points where the real and imaginary parts of the derivative separately go to zero. This phenomenon is illustrative of the numerous dimensions that characterize complex mappings.

A similar situation is seen with the Mean Value Theorem, which does not directly translate into the complex setting. The mean rate of change between two complex points does not ensure the existence of a point with a derivative that is equal to that of the given points. Conversely, Flett's and Myers' Theorems emerge as more suitable generalizations, exhibiting both theoretical soundness and practical relevance in specific analytic contexts. It is noteworthy that when the average rate is zero, Rolle's Theorem becomes a special case of the generalized Mean Value Theorem (GMVT), emphasizing a more profound connection between these two theorems within the context of holomorphic constraints.

The use of formal proofs and illustrative examples ($f(z) = z^2 + iz$) confirmed the validity of these generalizations in typical domains, emphasizing the structured behavior of holomorphic functions.

B. Conclusion

Holomorphic functions are equivalent to analytic functions in complex functions, holomorphic functions are functions that have derivatives at every point on the neighborhood and satisfy the Riemann Cauchy equation where the first partial derivatives are $u_x = v_y$ and $u_y = -v_x$. The properties of holomorphic functions are used in the complex domain. One of them is to extend the validity of the Mean Value Theorem and Rolle's Theorem to holomorphic functions. It is important to note that classical Rolle's and the Mean Value Theorems (MVT) are not universally valid for holomorphic functions.Theorems such as Flett's and Myers' provide structurally sound generalizations.

The connection between the Mean Value Theorem and the Rolle's Theorem can be proven from the equivalence property which implies that if the Mean Value Theorem holds, then the Rolle's Theorem also holds, so that the symmetry property of both is fulfilled. The equivalence between the Mean Value Theorems (MVT) and the Rolle's Theorem is more significant in the field of complex analysis due to the inherent properties of holomorphicity. The findings establish a more precise conceptual foundation for the application of real-variable intuition within complex-variable contexts.

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