

Downhill Product Connectivity Indices of Graphs

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ARTICLE INFO	ABSTRACT
<p>Published Online: 21 May 2025</p> <p>Corresponding Author: V.R. Kulli</p>	<p>In this study, we introduce the downhill product connectivity index and reciprocal downhill product connectivity index and their corresponding exponentials of a graph. Furthermore, we compute these indices for some standard graphs, wheel graphs.</p>
<p>KEYWORDS: downhill product connectivity index, reciprocal downhill product connectivity index, graph.</p>	

I. INTRODUCTION

In this study, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G . The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u . Any undefined terminologies and notations may be found in [1].

A topological index is a numerical parameter mathematically derived from the graph structure. In Chemical Graph Theory, concerning the definition of the topological index on the molecular graph and concerning chemical properties of drugs can be studied by the graph index calculation. Several topological indices have been considered in Theoretical Chemistry and many topological indices were defined by using vertex degree concept [2]. The Zagreb, Gourava, Revan, Sombor indices are the most degree based topological indices in Chemical Graph Theory, see [3-18]. Topological indices have their applications in various disciplines in Science and Technology [19-21].

A u - v path P in G is a sequence of vertices in G , starting with u and ending at v , such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \dots, v_{k+1}$ in G is a downhill path if for every $i, 1 \leq i \leq k, d_G(v_i) \geq d_G(v_{i+1})$.

A vertex v is downhill dominates a vertex u if there exists a downhill path originated from u to v . The downhill neighborhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u: v \text{ downhill dominates } u\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbors of v [22].

Recently, some downhill indices were studied in [23-28].

We introduce the downhill product connectivity index of a graph and it is defined as

$$DWP(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}}$$

Considering the downhill product connectivity index, we introduce the downhill product connectivity exponential of a graph G and defined it as

$$DWP(G, x) = \sum_{uv \in E(G)} x^{\frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}}}$$

We define the reciprocal downhill product connectivity index of a graph G as

$$RDWP(G) = \sum_{uv \in E(G)} \sqrt{d_{dn}(u)d_{dn}(v)}$$

Considering the reciprocal downhill product connectivity index, we introduce the reciprocal downhill product connectivity exponential of a graph G and defined it as

$$RDWP(G, x) = \sum_{uv \in E(G)} x^{\sqrt{d_{dn}(u)d_{dn}(v)}}$$

In this paper, the downhill product connectivity index, reciprocal downhill product connectivity index and their corresponding exponentials of certain graphs are computed.

II. RESULTS FOR SOME STANDARD GRAPHS

Proposition 1. Let G be r -regular with n vertices and $r \geq 2$. Then

$$DWP(G) = \frac{nr}{2(n-1)}.$$

Proof: Let G be an r -regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n - 1$ for every v in G .

$$\begin{aligned} DWP(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}} \\ &= \frac{nr}{2} \frac{1}{\sqrt{(n-1) \times (n-1)}} = \frac{nr}{2(n-1)}. \end{aligned}$$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$DWP(C_n) = \frac{n}{(n-1)}.$$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$DWP(K_n) = \frac{n}{2}.$$

Proposition 2. Let G be r -regular with n vertices and $r \geq 2$. Then

$$DWP(G, x) = \frac{nr}{2} x^{\frac{1}{(n-1)}}.$$

Proof: Let G be an r -regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n - 1$ for every v in G .

$$\begin{aligned} DWP(G, x) &= \sum_{uv \in E(G)} x^{\frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}}} \\ &= \frac{nr}{2} x^{\frac{1}{\sqrt{(n-1) \times (n-1)}}} = \frac{nr}{2} x^{\frac{1}{(n-1)}}. \end{aligned}$$

Corollary 2.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$DWP(C_n, x) = nx^{\frac{1}{(n-1)}}.$$

Corollary 2.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$DWP(K_n, x) = \frac{n(n-1)}{2} x^{\frac{1}{(n-1)}}.$$

Proposition 3. Let G be r -regular with n vertices and $r \geq 2$. Then

$$RDWP(G) = \frac{n(n-1)r}{2}.$$

Proof: Let G be an r -regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n - 1$ for every v in G .

$$\begin{aligned} RDWP(G) &= \sum_{uv \in E(G)} \sqrt{d_{dn}(u)d_{dn}(v)} \\ &= \frac{nr\sqrt{(n-1) \times (n-1)}}{2} = \frac{n(n-1)r}{2}. \end{aligned}$$

Corollary 3.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$RDWP(C_n) = n(n-1).$$

Corollary 3.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$RDWP(K_n) = \frac{n(n-1)^2}{2}.$$

Proposition 4. Let G be r -regular with n vertices and $r \geq 2$. Then

$$RDWP(G, x) = \frac{nr}{2} x^{n-1}.$$

Proof: Let G be an r -regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n - 1$ for every v in G .

$$\begin{aligned} RDWP(G, x) &= \sum_{uv \in E(G)} x^{\sqrt{d_{dn}(u)d_{dn}(v)}} \\ &= \frac{nr}{2} x^{\sqrt{(n-1) \times (n-1)}} = \frac{nr}{2} x^{n-1}. \end{aligned}$$

Corollary 4.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$RDWP(C_n, x) = nx^{n-1}.$$

Corollary 4.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$RDWP(K_n, x) = \frac{n(n-1)}{2} x^{n-1}.$$

III. RESULTS FOR WHEEL GRAPHS

The wheel W_n is the join of C_n and K_1 . Clearly W_n has $n+1$ vertices and $2n$ edges. The vertex K_1 is called apex and the vertices of C_n are called rim vertices.

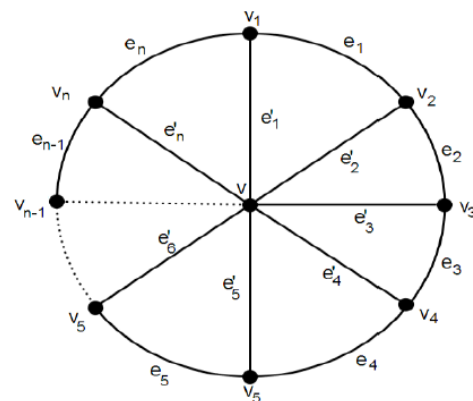


Figure 1. Wheel W_n

Lemma 1. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then there are two types of the downhill degree of vertices as given below:

$$V_1 = \{u \in V(W_n) \mid d_{dn}(u) = n\}, \quad |V_1| = 1.$$

$$V_2 = \{u \in V(W_n) \mid d_{2dn}(u) = n-1\}, \quad |V_2| = n.$$

Lemma 2. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then there are two types of the downhill degree of edges as follows:

$$E_1 = \{uv \in E(W_n) \mid d_{dn}(u) = n, d_{dn}(v) = n-1\}, \quad |E_1| = n.$$

$$E_2 = \{uv \in E(W_n) \mid d_{dn}(u) = d_{dn}(v) = n-1\}, \quad |E_2| = n.$$

Theorem 1. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then the downhill product connectivity index of W_n is

$$DWP(W_n) = \frac{n}{\sqrt{n(n-1)}} + \frac{n}{n-1}.$$

Proof: From definition and by Lemma 2, we deduce

$$DWP(W_n) = \sum_{uv \in E(W_n)} \frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}}$$

$$= \frac{n}{\sqrt{n \times (n-1)}} + \frac{n}{\sqrt{(n-1) \times (n-1)}}$$

$$= \frac{n}{\sqrt{n(n-1)}} + \frac{n}{n-1}.$$

Theorem 2. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then the downhill product connectivity exponential of W_n is

$$DWP(W_n, x) = nx^{\frac{1}{\sqrt{n(n-1)}}} + nx^{\frac{1}{n-1}}.$$

Proof: From definition and by Lemma 2, we deduce

$$DWP(W_n, x) = \sum_{uv \in E(W_n)} x^{\frac{1}{\sqrt{d_{dn}(u)d_{dn}(v)}}}$$

$$= nx^{\frac{1}{\sqrt{n(n-1)}}} + nx^{\frac{1}{\sqrt{(n-1)(n-1)}}}$$

$$= nx^{\frac{1}{\sqrt{n(n-1)}}} + nx^{\frac{1}{n-1}}.$$

Theorem 3. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then the reciprocal downhill product connectivity index of W_n is

$$RDWP(W_n) = n\sqrt{n(n-1)} + n(n-1).$$

Proof: From definition and by Lemma 2, we deduce

$$RDWP(W_n) = \sum_{uv \in E(W_n)} \sqrt{d_{dn}(u)d_{dn}(v)}$$

$$= n\sqrt{n(n-1)} + n\sqrt{(n-1)(n-1)}$$

$$= n\sqrt{n(n-1)} + n(n-1).$$

Theorem 4. Let W_n be a wheel with $n+1$ vertices, $n \geq 4$. Then the reciprocal downhill product connectivity exponential of W_n is

$$RDWP(W_n, x) = nx^{\sqrt{n(n-1)}} + nx^{(n-1)}.$$

Proof: From definition and by Lemma 2, we deduce

$$RDWP(W_n, x) = \sum_{uv \in E(W_n)} x^{\sqrt{d_{dn}(u)d_{dn}(v)}}$$

$$= nx^{\sqrt{n(n-1)}} + nx^{\sqrt{(n-1)(n-1)}}$$

$$= nx^{\sqrt{n(n-1)}} + nx^{(n-1)}.$$

IV. CONCLUSION

In this paper, the downhill product connectivity index and reciprocal downhill product connectivity index are defined. These indices and their corresponding exponentials for some standard graphs and wheel graphs are determined.

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