International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 13 Issue 04 April 2025, Page no. – 5138-5143
Index Copernicus ICV: 57.55, Impact Factor: 8.615
DOI: 10.47191/ijmcr/v13i4.17

# ON THREE CONSECUTIVE PAIRS OF LUCAS AND FIBONACCI NUMBERS CONNECTED TO PELL-LUCAS AND PELL POLYNOMIALS

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Abstract. In the present paper we connect three consecutive pairs of Lucas and Fibonacci numbers, namely, $\{(L_{3n}, F_{3n}), (L_{3n+1}, F_{3n+1}), (L_{3n+2}, F_{3n+2}) : n = 0, 1, 2, 3, \cdots\}$ to Pell-Lucas and Pell polynomials, namely,

$$P_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2+1}\right)^n + \left(x - \sqrt{x^2+1}\right)^n\right] \text{ and}$$
$$Q_n(x) = \frac{1}{2\sqrt{x^2+1}}\left[\left(x + \sqrt{x^2+1}\right)^n - \left(x - \sqrt{x^2+1}\right)^n\right].$$

We workout some interesting combinatorial identities of $X_n = P_n(2), Y_n = Q_n(2)$ and the three consecutive pairs of Lucas and Fibonacci numbers.

*2020 Mathematics Subject Classification.* 05A19, 11B39.
*Keywords.* Combinatorial identities, Generalized Lucas - Fibonacci pairs, Tchebyshev polynomials

## 1. Introduction

In [1], a pair of sequences $(x_n, y_n)$ is introduced by evaluating the well known Tchebyshev polynomials of first and second kind, namely,

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right] \text{ and}$$
$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x + \sqrt{x^2-1}\right)^{n+1} - \left(x - \sqrt{x^2-1}\right)^{n+1}\right],$$
$$n = 0, 1, 2, 3, \cdots \text{ at } x = 9.$$

The six consecutive pairs of Lucas and Fibonacci numbers, namely, $\{(L_{6n+k}, F_{6n+k}) : k = 0, 1, 2, 3, 4, 5$ and $n = 0, 1, 2, 3, \cdots\}$ have beautiful connection to $(x_n, y_n)$ given by

$$L_{6n+k} = L_k x_n + 20 F_k y_n$$
$$F_{6n+k} = F_k x_n + 4 L_k y_n,$$
$$k = 0, 1, 2, 3, 4, 5 \text{ and } n = 0, 1, 2, 3, \cdots .$$

Some interesting combinatorial identities of $(x_n, y_n)$ and $(L_{6n+k}, F_{6n+k})$ are presented there [1].

Motivated strongly by the above paper, first we discovered one more simple and beautiful connection between $(L_{3n}, F_{3n})$ and $(P_n(2), Q_n(2))$, namely,

$$\begin{aligned} L_{3n} &= \left(2+\sqrt{5}\right)^n + \left(2-\sqrt{5}\right)^n \\ &= \left(2+\sqrt{2^2+1}\right)^n + \left(2-\sqrt{2^2+1}\right)^n \\ &= 2\ P_n(2) \end{aligned}$$

and

$$\begin{aligned} F_{3n} &= \frac{1}{\sqrt{5}}\left[\left(2+\sqrt{5}\right)^n - \left(2-\sqrt{5}\right)^n\right] \\ &= \frac{1}{\sqrt{2^2+1}}\left[\left(2+\sqrt{2^2+1}\right)^n - \left(2-\sqrt{2^2+1}\right)^n\right] \\ &= 2\ Q_n(2). \end{aligned}$$

We introduce, $X_n = P_n(2)$ and $Y_n = Q_n(2)$. Then they fit so well that one may derive

$$\begin{aligned} &L_{3n+k} = L_k\ X_n + 5\ F_k\ Y_n \\ &F_{3n+k} = F_k\ X_n + L_k\ Y_n \\ &\quad k = 0,1,2,3 \text{ and } n = 0,1,2,\cdots. \end{aligned}$$

In the present paper, we derive some interesting combinatorial identities of $(X_n, Y_n)$ and $(L_{3n+k}, F_{3n+k})$. The study of combinatorial identities have many interesting facets and applications available in the recent literature. One csn find combinatorial identities of the following entities-

(i) Catalan numbers [2,3]
(ii) Lucas and Fibonacci Numbers [1,7,10,11,13,15]
(iii) Tchebyshev Polynomials [1,3–6,8]
(iv) Pell - Lucas and Pell polynomials [8,14]
(v) Brahmagupta polynomials [9,12]

The number theory of $L_n$ and $F_n$ enables one to derive many Combinatorial identities mentioned in the end of the above paragraph.

## 2. Recurrence Relations

For the sake of smooth computation of $\{(L_{3n+k}, F_{3n+k}) : k = 0,1,2,3 \textit{ and } n = 0,1,2,3, .....\}$, a pair of sequences, namely $(X_n, Y_n)$ described by Pell-Lucas and Pell polynomials is extensively applied.

**Definition:**

$$X_n = P_n(2),\ Y_n = Q_n(2) \qquad n = 0,1,2\cdots.$$

The pair $(X_n, Y_n)$ has the following binet form :

$$\begin{aligned} X_n &= \frac{1}{2}\left[\zeta^n + \eta^n\right] \\ Y_n &= \frac{\zeta^n - \eta^n}{\zeta - \eta} \end{aligned}$$

where $\zeta = 2+\sqrt{5}$, $\eta = 2-\sqrt{5}$ and $n = 0,1,2\cdots$.
$\zeta$ and $\eta$ satisfy the following relations : $\zeta+\eta = 4$, $\zeta-\eta = 2\sqrt{5}$, $\zeta\ \eta = -1$. By applying above binet form, one can show that $(X_n, Y_n)$ satisfy the following relations which are directed by Pell-Lucas and Pell polynomials [8,14].

**Identities 2.1**

$$\begin{aligned} (1)\ X_{n+1} &= 4\ X_n + X_{n-1} \\ &= \frac{Y_{n+2}+Y_n}{2} \\ &= 2\ X_n + 5\ Y_n \\ (2)\ Y_{n+1} &= 4\ Y_n + Y_{n-1} \\ &= \frac{X_{n+2}+X_n}{10} \\ &= X_n + 2\ Y_n \end{aligned}$$

$$n = 1,2,3\ldots$$

**Identities 2.2**

$$\begin{aligned}
(1)\ 2\ L_{3n+k} &= L_{3n}\ L_k + 5\ F_{3n}\ F_k \\
(2)\ 2\ F_{3n+k} &= L_{3n}\ F_k + F_{3n}\ L_k \\
(3)\ L_{3n+k} &= L_k\ X_n + 5\ F_k\ Y_n \\
&= (L_k - 2\ F_k)\ X_n + F_k\ X_{n+1} \\
&= (5\ F_k - 2\ L_k)\ Y_n + L_k\ Y_{n+1} \\
(4)\ F_{3n+k} &= F_k\ X_n + L_k\ Y_n \\
&= \frac{(5\ F_k - 2\ L_k)\ X_n + L_k\ X_{n+1}}{5} \\
&= (L_k - 2\ F_k)\ Y_n + F_k\ Y_{n+1} \\
k = 0,1,2,3\ and\ n = 0,1,2,\ldots
\end{aligned}$$

**Identities 2.3**

$$\begin{aligned}
(1)\ L_{3(n+1)+k} &= 4\ L_{3n+k} + L_{3(n-1)+k} \\
(2)\ F_{3(n+1)+k} &= 4\ F_{3n+k} + F_{3(n-1)+k} \\
k = 3,4,5\ and\ n = 1,2,3,\ldots
\end{aligned}$$

**Identities 2.4**

$$\begin{aligned}
(1)\ 2\ L_{3n+1} &= L_{3n+3} - L_{3n} \\
(2)\ 2\ F_{3n+1} &= F_{3n+3} - F_{3n} \\
(3)\ 2\ L_{3n+2} &= L_{3n+3} + L_{3n} \\
(4)\ 2\ F_{3n+2} &= F_{3n+3} + F_{3n} \\
(5)\ L_{3n+3} &= 2\ L_{3n} + 5\ F_{3n} \\
(6)\ F_{3n+3} &= L_{3n} + 2\ F_{3n} \\
(7)\ 2\ L_{3n+4} &= 3\ L_{3n+3} + L_{3n} \\
(8)\ 2\ F_{3n+4} &= 3\ F_{3n+3} + F_{3n} \\
(9)\ 2\ L_{3n-1} &= L_{3n+3} - 3\ L_{3n} \\
(10)\ 2\ F_{3n-1} &= F_{3n+3} - 3\ F_{3n} \\
n = 1,2,3,\ldots
\end{aligned}$$

## 3. Matrix power identities, Cassini determinant identities and Generating functions

In the literature, a matrix power identity called Brahmagupta power identity ( [9], [12]) is described

$$\begin{bmatrix} \alpha_n & \beta_n \\ t\ \beta_n & \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ t\ \beta & \alpha \end{bmatrix}^n$$

for

$$\begin{aligned}
\alpha_n &= \frac{1}{2}\left[\left(\alpha + \beta\sqrt{t}\right)^n + \left(\alpha - \beta\sqrt{t}\right)^n\right] \\
\beta_n &= \frac{1}{2\sqrt{t}}\left[\left(\alpha + \beta\sqrt{t}\right)^n - \left(\alpha - \beta\sqrt{t}\right)^n\right]
\end{aligned}$$

When $\alpha = 2$, $\beta = 1$ and $t = 5$ we obtain $X_n$ and $Y_n$. As a consequence we have the following matrix power identities and Cassini determinant identities described in the following results

**Identities 3.1**

$$(1) \qquad \begin{bmatrix} X_n & Y_n \\ 5\ Y_n & X_n \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^n$$

$$(2) \qquad \begin{bmatrix} Y_{n-1} & Y_n \\ Y_n & Y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{n-1}$$

$$(3) \qquad \begin{bmatrix} X_{n-1} & X_n \\ X_n & X_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{n-1}$$

$$n = 1, 2, 3, \ldots$$

**Identities 3.2**

$$(1) \qquad X_n^2 - 5\ Y_n^2 = (-1)^n$$

$$(2) \qquad Y_{n-1}\ Y_{n+1} - Y_n^2 = (-1)^n$$

$$(3) \qquad X_{n-1}\ X_{n+1} - X_n^2 = (-1)^{n-1}\ 5$$

$$n = 1, 2, 3, \ldots$$

**Identities 3.3**

$$\begin{bmatrix} L_{3n+k} & F_{3n+k} \\ 5\ F_{3n+k} & L_{3n+k} \end{bmatrix} = \begin{bmatrix} L_k & F_k \\ 5\ F_k & L_k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^n$$

$$k = 0, 1, 2, 3 \ and \ n = 0, 1, 2, 3, \ldots$$

Next we state the generating function of $X_n$, $Y_n$, $L_{3n+k}$ and $F_{3n+k}$, $k = 0, 1, 2, 3$ with the help of the identities given by (2.1) and (2.2).

**Identities 3.4**

$$(1) \qquad \sum_{n=0}^{\infty} L_{3n+k}\ t^n = \frac{L_k + (L_{k+3} - 4\ L_k)\ t}{1 - 4t - t^2}$$

$$(2) \qquad \sum_{n=0}^{\infty} F_{3n+k}\ t^n = \frac{F_k + (F_{k+3} - 4\ F_k)\ t}{1 - 4t - t^2}$$

$$(3) \qquad \sum_{n=0}^{\infty} X_n\ t^n = \frac{1 - 2\ t}{1 - 4\ t - t^2}$$

$$(4) \qquad \sum_{n=0}^{\infty} Y_n\ t^n = \frac{t}{1 - 4t - t^2}$$

$$k = 0, 1, 2, 3.$$

## 4. Summation identities

By suitable rearranging recurrence relations one like below the summation identities can be derived non trivially.

$$X_k \quad = \quad 4\ X_{k-1} + X_{k-2},\ k = 2, 3, ...., n$$

Adding all and simplifying we get the following summation identities:

**Identities 4.1**

$$(1) \qquad 4 \sum_{k=0}^{n} X_k = X_{n+1} + X_n + 1$$

$$(2) \qquad 4 \sum_{k=0}^{n} Y_k = Y_{n+1} + Y_n - 1$$

$$n = 0, 1, 2, 3, \ldots$$

Following the same ideas, one can derive the following summation identities :

**Identities 4.2**

$$(1) \qquad 4\sum_{k=0}^{n} X_{2k} = X_{2n+1} + 2$$

$$(2) \qquad 4\sum_{k=0}^{n} Y_{2k} = Y_{2n+1} - 1$$

$$n = 0, 1, 2, 3, \ldots$$

The binet form of $X_n$ and $Y_n$ will guide the following square sum identities:
**Identities 4.3**

$$(1) \qquad \sum_{k=0}^{n} X_k^2 = \frac{1}{32}\left[X_{2n+2} - X_{2n}\right] + \left[\frac{2+(-1)^n}{4}\right]$$

$$(2) \qquad \sum_{k=0}^{n} Y_k^2 = \frac{1}{160}\left[(X_{2n+2} - X_{2n}\right] - \frac{(-1)^n}{20}$$

$$n = 0, 1, 2, 3, \ldots$$

By making use of $L_{3n+k} = L_k\ X_n + 5\ F_k\ Y_n$ and $F_{3n+k} = F_k\ X_n + L_k\ Y_n : k = 0, 1, 2, 3$. One can directly derive the following square sum identities :
**Identities 4.4**

$$(1) \qquad \sum_{r=0}^{n} L_{3r+k}^2 = \frac{L_{2k}}{80}\left[X_{2n+2} - X_{2n}\right] + \frac{F_{2k}}{4}\left[Y_{2n+1} - 1\right] + \frac{F_k^2}{4} - \frac{(-1)^{n+k}}{5}$$

$$(2) \qquad \sum_{r=0}^{n} F_{3r+k}^2 = \frac{L_{2k}}{16}\left[X_{2n+2} - X_{2n}\right] + \frac{5\ F_{2k}}{4}\left[Y_{2n+1} - 1\right] + \frac{L_k^2}{4} + (-1)^{n+k}$$

$$k = 0, 1, 2, 3 \ and\ n = 0, 1, 2, 3, \ldots$$

## 5. Convolution identities

The binet forms will guide the following convolution identities :
**Identities 5.1**

$$(1) \qquad \sum_{k=0}^{n} X_k\ X_{n-k} = \frac{1}{2}\left[(n+1)X_n + Y_{n+1}\right]$$

$$(2) \qquad \sum_{k=0}^{n} Y_k\ Y_{n-k} = \frac{1}{10}\left[(n+1)X_n - Y_{n+1}\right]$$

$$(3) \qquad \sum_{k=0}^{n} X_k\ Y_{n-k} = \sum_{k=0}^{n} Y_k\ X_{n-k} = \left(\frac{n+1}{2}\right)\ Y_n$$

$$n = 0, 1, 2, \ldots$$

**Identities 5.2**

$$(1) \qquad \sum_{r=0}^{n} L_{3r+k}\ L_{3(n-r)+k} = (n+1)\ L_{3n+2k} + (-1)^k\ 2\ Y_{n+1}$$

$$(2) \qquad \sum_{r=0}^{n} F_{3r+k}\ F_{3(n-r)+k} = \frac{1}{5}\left[(n+1)\ L_{3n+2k} - (-1)^k\ 2\ Y_{n+1}\right]$$

$$(3) \qquad \sum_{r=0}^{n} L_{3r+k}\ F_{3(n-r)+k} = \sum_{r=0}^{n} F_{3r+k}\ L_{3(n-r)+k} = (n+1)F_{3n+2k}$$

$$k = 0, 1, 2, 3 \ and\ n = 0, 1, 2, 3, \ldots$$

The identities 5.1 and 5.2 can be derived directly using the identities of the section (2). For a good account of convolution identities one may refer to [2].

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