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Hasse diagrams of basic blocks containing four comparable reducible elements and having nullity four

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Abstract

In 2020, Bhavale and Waphare introduced the concepts of fundamental basic block and basic block, associated to dismantlable lattices. Further, they have provided the recursive formulae of the number of non-isomorphic fundamental basic blocks as well as basic blocks, containing r comparable reducible elements and having nullity l. In this paper, we actually obtain the Hasse diagrams of the basic blocks containing four comparable reducible elements and having nullity four.

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1 Introduction

Birkhoff [1] in 1940 raised the open problem of enumerating all posets/lattices which are uniquely determined up to isomorphism by their diagrams, considered purely as graphs. Many authors all over the world attempted this problem. In 1979, Kyuno [8], gave an inductive algorithm to construct finite lattices, wherein he could obtain Hasse diagrams of the lattices on up to 9 elements, up to isomorphism. Recently, Monteiro et al. [9] obtained Hasse diagrams of posets with up to 7 elements, and the number of posets with 10 elements, without the use of computer programs. For more details on enumerations, reader may see [2] and [3].

In 2020, Bhavale and Waphare [4] introduced the concepts of fundamental basic block and basic block, associated to dismantlable lattices. Further, they have provided the recursive formulae of the number of non-isomorphic fundamental basic blocks as well as basic blocks, containing r comparable reducible elements and having nullity l. In this paper, we actually obtain the Hasse diagrams of the basic blocks containing four comparable reducible elements and having nullity four.

An element a in a lattice L is meet-reducible(join-reducible) in L if there exist $b, c \in L$ both distinct from a, such that $\inf\{b, c\} = a$ ($\sup\{b, c\} = a$). An element a in a lattice L is said to be reducible if it is either meet-reducible or join-reducible. a is said to be meet-irreducible(join-irreducible) if it is not meet-reducible(join-reducible). a is doubly irreducible if it is both meet-irreducible and join-irreducible.

A finite lattice of order n is called dismantlable if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i (see [5]). An element x of a poset P is said to be doubly irreducible if it has at most one upper cover and at most one lower cover in P. Let $Irr(P)$ denotes the set of all doubly irreducible elements of P. The nullity of a poset P is defined as the nullity of its cover graph. A poset P is a basic block if it is one element or $Irr(P) = \emptyset$ or removal of any doubly irreducible element reduces nullity by one (see [4]). A dismantlable lattice F is said to be a fundamental basic block if it is a basic block and and all the adjunct pairs in the adjunct representation of F into chains are distinct(see [4]). For the other definitions, notation and terminology; reader may refer [6] and [7].

2 Counting of basic blocks

Let $\mathscr{F}_r(l)$ be the class of all non-isomorphic fundamental basic blocks containing r comparable reducible elemnets and having nullity l. Note that $\mathscr{F}_0(0)$ consists of a 1-chain only. The following results are due to Bhavale and Waphare [4].

Theorem 1.1[4]: For fixed $r \geq 1$, and for $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,

$$|\mathscr{F}_{r+1}(l)| = \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)|.$$

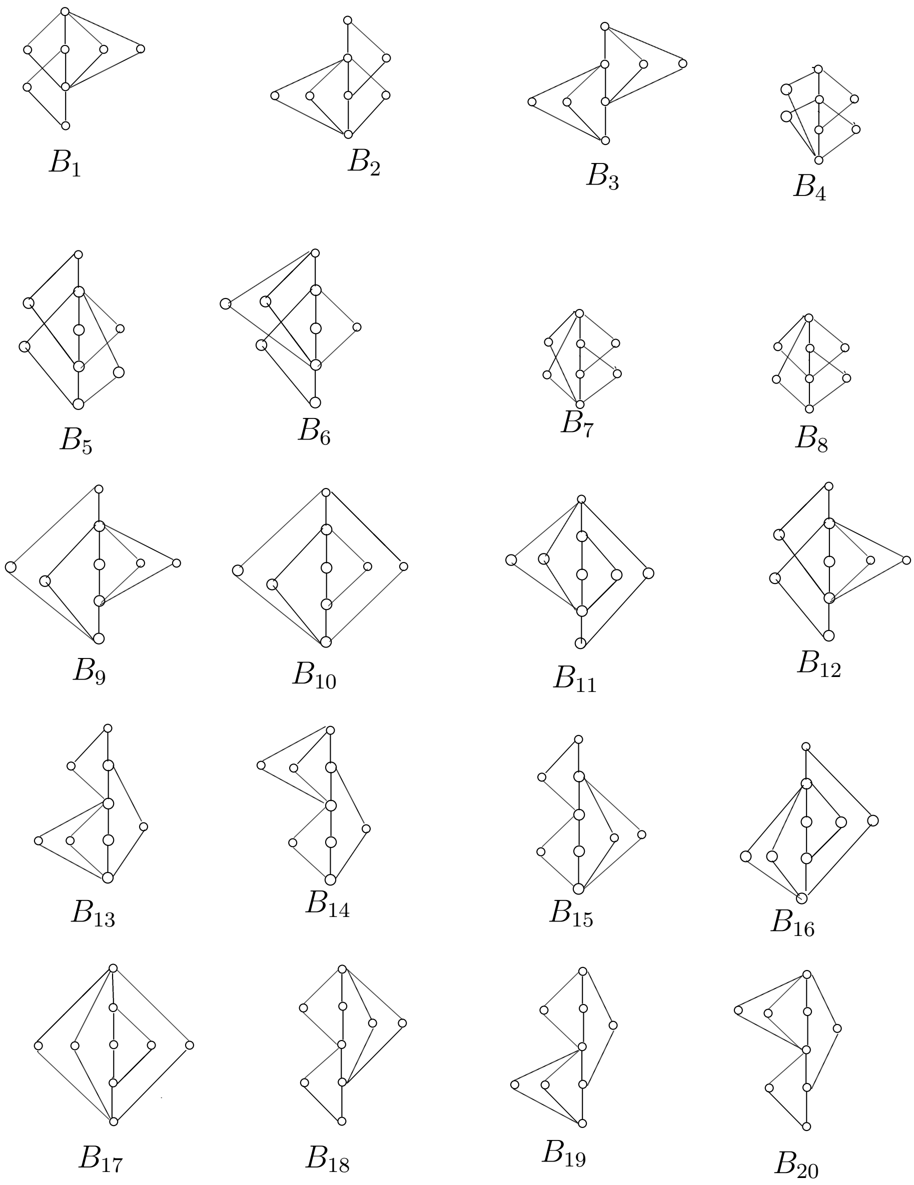
Using Theorem 1.1, we have $|\mathscr{F}_4(2)| = 3$, $|\mathscr{F}_4(3)| = 16$, and $|\mathscr{F}_4(4)| = 15$.
Let $\mathscr{B}_r(l)$ be the class of all non-isomorphic basic blocks containing r comparable reducible elemnets and having nullity l. Note that $\mathscr{B}_0(0)$ consists of a 1-chain only.

Theorem 1.2[4]: For fixed $r \geq 2$, and for $[\frac{r+1}{2}] \leq m \leq l \leq \binom{r+1}{2}$,
$|\mathscr{B}_r(l)| = \sum_{m=[\frac{r+1}{2}]}^{l} \binom{l-1}{m-1} |\mathscr{F}_r(m)|.$

Using Theorem 1.2, $|\mathscr{B}_4(4)| = (3)(3) + (3)(16) + (1)(15) = 9 + 48 + 15 = 72$.

3 Hasse diagrams of basic blocks

In this section, we provide up to isomorphism, Hasse diagrams of all 72 basic blocks which are the lattices containing four comparable reducible elements and having nullity four.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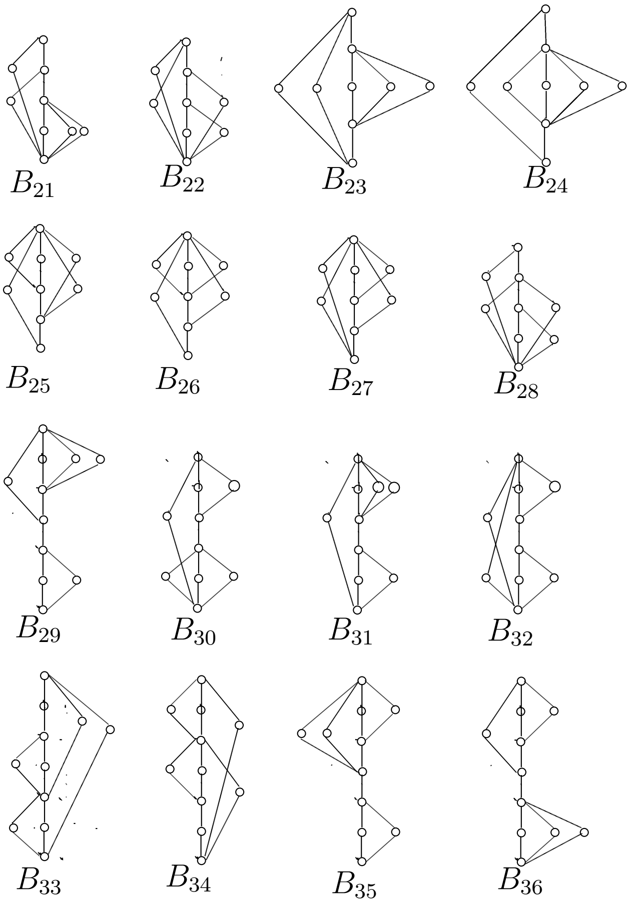
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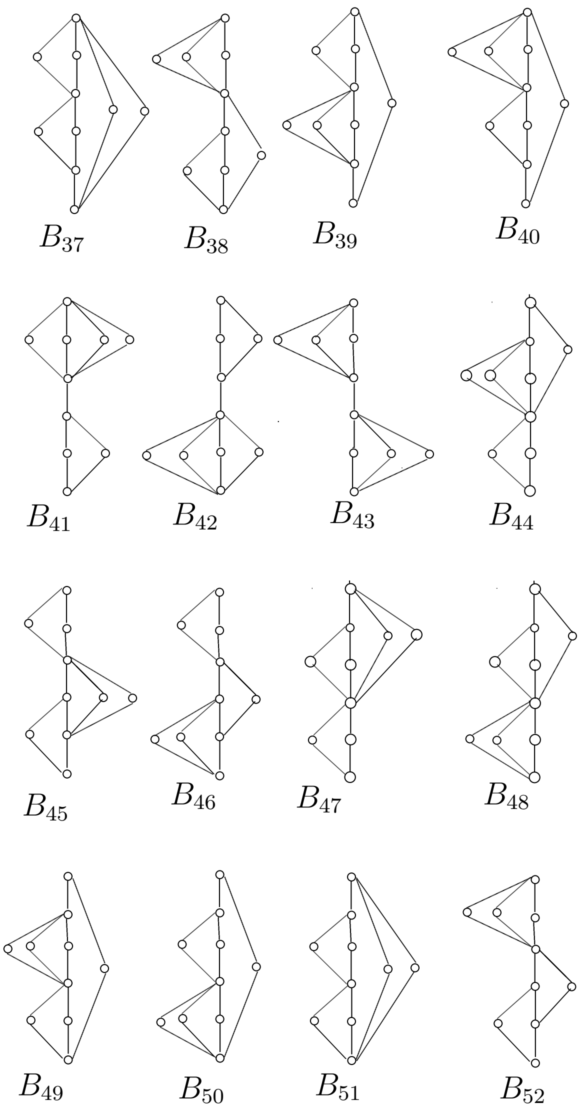
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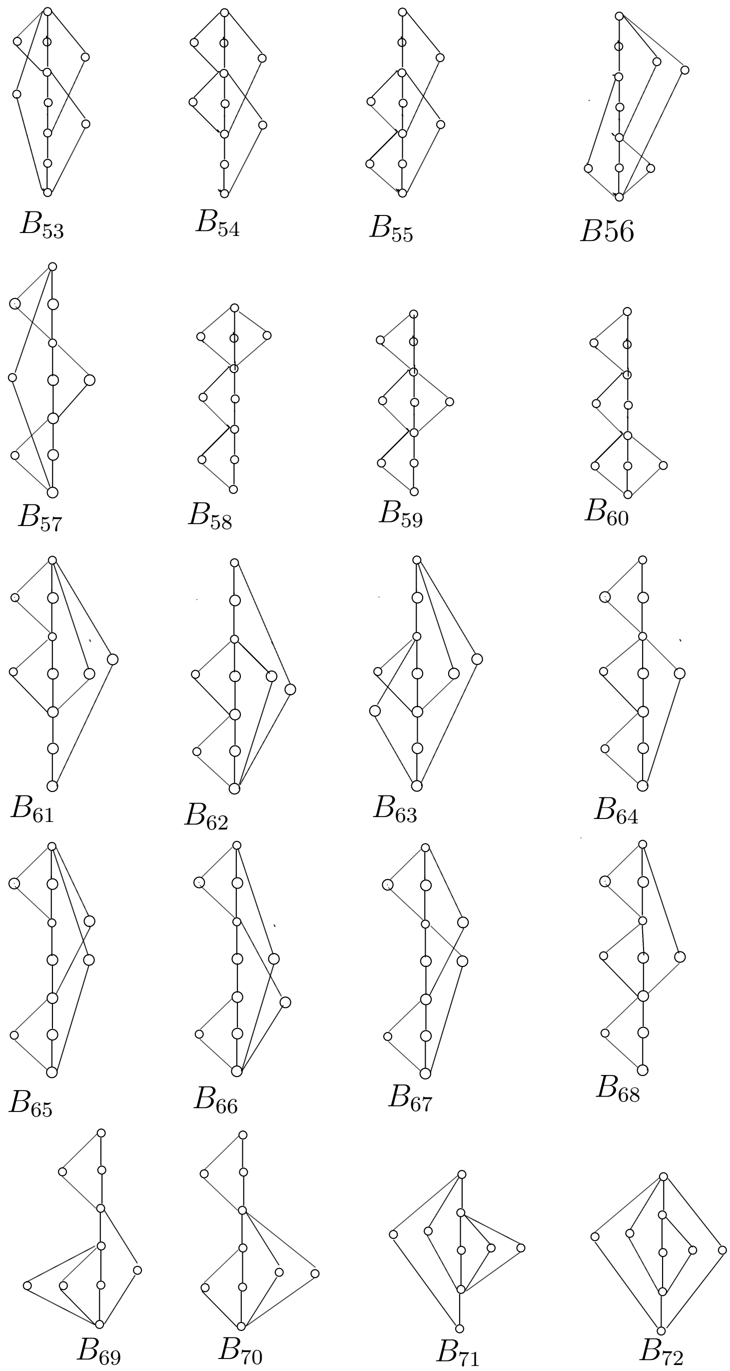

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