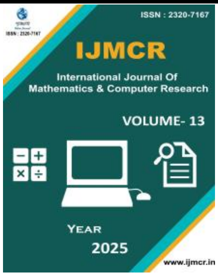
International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 13 Issue 04 April 2025, Page no. – 5096-5098

Index Copernicus ICV: 57.55, Impact Factor: 8.615

DOI: 10.47191/ijmcr/v13i4.11



Stability Analysis of the Improved Analytic Solution of the Mathematical Model for the Complications of Diabetes Mellitus

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ARTICLE INFO

Published Online:
19 April 2025

ABSTRACT

Diabetes mellitus has become a global silent epidemic rising at an alarming rate throughout the world due to increase in life expectancy, obesity and sedentary lifestyles. It is a chronic disease that can lead to complications over time. These complications can be prevented or minimized with the support of government, private sectors and medical expertise.

This paper proposed an improvement in the existing mathematical model based on number of diabetics with complications and number of diabetics with and without complications. Stability of the proposed model is analyzed and observed that the complications can be controlled during diabetes persists. Analytic solution of the system of ordinary differential equation is achieved.

In this paper, the proposed model precise the understanding of diabetes, its complications and how it can be controlled. The stability analysis of the model showed that the models were stable asymptotically at various parameter values. The model showed that the rate at which complications are controlled is very important parameter in controlling the diabetes and its complications. If the value of the rate at which complications are controlled is high, then the number of diabetics with complications is decreased. Numerical solution of the proposed models were obtained using Euler method. Numerical simulations of the analytic solutions were obtained at various values of the parameters.

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KEYWORDS: Mathematical Model, Stability, Analytic Solution, Diabetes Mellitus

I. INTRODUCTION

Diabetes mellitus has become a global silent epidemic rising at an alarming rate throughout the world due to increase in life expectancy, obesity and sedentary lifestyles. According to World Health Organization (2019), approximately 463 million adults were living with diabetes; this will rise to 700 million by 2045[15]. According to International Diabetes Foundation, diabetes caused 4.2 million deaths[10]. According to the study of Indian Council of Medical Research and collaborating institutions, the first to cover entire states has shown that 13 people in 100 have diabetes in urban Jharkhand but just 3 per 100 in the state's rural areas.

Mathematical modelling is a process to convert real life problems into mathematical basically it is a description of a system using mathematical concepts and languages. A model help to explain a system and to study the effects of different components and to make predictions about behaviour.

II. MATERIALS AND METHODS

In the previous paper, Akinsola and Oluyo (2014) ,proposed a mathematical model of systems of differential equations for the complications and control of diabetes mellitus with initial conditions as follows[1]

$$C'(t) = I - (\rho + \theta)C(t) + \rho N(t), C(0) = C_0 \quad (1)$$

$$N'(t) = 2I - (\nu + \delta)C(t) - \mu N(t), N(0) = N_0 \quad (2)$$

which is represented in matrix form as

$$\begin{pmatrix} C(t) \\ N(t) \end{pmatrix}' = \begin{pmatrix} -(\rho + \theta) & \rho \\ -(\nu + \delta) & -\mu \end{pmatrix} \begin{pmatrix} C(t) \\ N(t) \end{pmatrix} + \begin{pmatrix} I \\ 2I \end{pmatrix}, \begin{pmatrix} C_0 \\ N_0 \end{pmatrix} \quad (3)$$

The analytic solution of the resulting system of equation obtained using elimination method with un determined coefficient in Akinsola and Oluyo (2014) was

$$C(t) = K_1 e^{-\eta_1 t} + K_2 e^{-\eta_2 t} + \frac{\alpha}{\beta} I \quad (4)$$

$N(t) = K_1 e^{-\eta_1 t} + K_2 e^{-\eta_2 t} + \frac{\alpha}{\beta} I + \frac{\theta}{\rho} K_1 e^{-\eta_1 t} + \frac{\theta}{\rho} K_2 e^{-\eta_2 t} + \frac{\theta\alpha}{\pi\beta} I - \frac{I}{\rho} - \frac{I}{\rho}(\eta_1 K_1 e^{-\eta_1 t} + \eta_2 K_2 e^{-\eta_2 t})$ (5)

This analytic equation is modified in this paper as follows:

$C(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + \frac{\alpha}{\beta} I$ (6)

$N(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + \frac{\alpha}{\beta} I + \frac{\theta}{\rho} K_1 e^{\lambda_1 t} + \frac{\theta}{\rho} K_2 e^{\lambda_2 t} + \frac{\theta\alpha}{\pi\beta} I - \frac{I}{\rho} - \frac{I}{\rho}(-\lambda_1 K_1 e^{\lambda_1 t} - \lambda_2 K_2 e^{\lambda_2 t})$ (7)

where $\theta = \gamma + \mu + \nu + \delta$ (8)

$\lambda_1 = -\eta_1 = \frac{1}{2}\left(\sigma - \sqrt{\sigma^2 - 4\beta}\right)$ (9)

$\lambda_2 = -\eta_2 = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 - 4\beta}\right)$ (10)

$\sigma = \rho + \theta + \mu$ (11)

$\beta = \rho\nu + \rho\delta + \rho\mu + \mu\theta$ (12)

$\alpha = 2\rho + \mu$ (13)

$K_1 = \frac{\beta(\rho+\theta+\lambda_2)C_0 + I(-\alpha\lambda_2 - \beta) - \rho\beta N_0}{\beta(\lambda_2 - \lambda_1)}$ (14)

$K_2 = \frac{-\beta(\rho+\theta+\lambda_1)C_0 + I(\beta - \alpha\lambda_1) + \rho\beta N_0}{\beta(\lambda_2 - \lambda_1)}$ (15)

A. *Stability Analysis*

We have to find the critical points at C'(t)=N'(t)=0

$$0 = I - (\rho + \theta)C(t) + \rho N(t)$$
$$0 = 2I - (\nu + \delta)C(t) - \mu N(t)$$

Solving these equations simultaneously and simplifying we obtained the critical points as follows:

$C^*(t) = \frac{(2\rho+\mu)I}{\nu\rho+\mu\theta+\rho\delta+\rho\mu}$ (16)

$N^*(t) = \frac{(2(\rho+\theta)-(\mu+\delta))I}{\nu\rho+\mu\vartheta+\rho\delta+\rho\mu}$ (17)

Next step is to find the eigen values, For this we have a matrix in equation (3),

So the characteristic equation of the matrix is,

$\chi^2 + (\rho + \theta + \mu)\chi + (\rho\nu + \rho\delta + \rho\mu + \mu\theta) = 0$ (18)

Let χ_1 and χ_2 be the roots of characteristic equations.

Now substituting equation (11) and (12) in (18) we obtained the quadratic equation in the form as follows:

$\chi^2 + \sigma\chi + \beta = 0$ (19)

Therefore the roots of this quadratic equation gives the eigenvalues of the characteristic equation. We have

$\chi_1 = \frac{-\sigma + \sqrt{\sigma^2 - 4\beta}}{2}$ (20)

$\chi_2 = \frac{-\sigma - \sqrt{\sigma^2 - 4\beta}}{2}$ (21)

Now substituting the values of equations (9) and (10) in equations (20) and (21) gives

$\chi_1 = \lambda_1$ (22)

$\chi_2 = \lambda_2$ (23)

Now let us consider a homogeneous system of first order differential equation of the form

$\frac{du}{dt} = u'(t) = a_1(t)u + b_1(t)v$ (24)

$\frac{dv}{dt} = v'(t) = a_2(t)u + b_2(t)v$ (25)

Where a_1, a_2, b_1, b_2 are real constants.

Therefore $det = a_1 b_2 - b_1 a_2 \neq 0$ (26)

Let χ_1 and χ_2 be the roots of quadratic equation then

$\chi^2 - (a_1 + b_2)\chi + (a_1 b_2 - a_2 b_1) = 0$ (27)

Equation (27) is called the characteristic equation or auxiliary equation and χ_1 and χ_2 are eigenvalues of the characteristic equation. Eigenvalues are the special set of scalar value λ associated with the set of linear equation which satisfy the equation

$$A \cdot x = \lambda \cdot x$$

Now let us say A is an "n x n" matrix and λ is an eigenvalues of matrix A, then x is a non- zero vector or eigenvector of A corresponding to eigenvalue λ.

Now the matrix in equation (3) can be written as a vector form as follows

$Y = A(t)Y + g(t);$

$Y(t_0) = Y_0$ (28)

Where A is a real n x n matrix.

Hence, we can conclude that

i. If the negative real parts and zero real parts of all the eigenvalues of matrix A are not complex then the solution of equation (28) $Y^* = 0$ is stable.
ii. The critical points of equation (28) is asymptotically stable if and only if all eigenvalues of matrix A contains non-positive real parts and the critical points of equation (28) is not stable if either one or more eigenvalues of A contains non-negative real parts.

III. RESULTS AND DISCUSSION

By taking limit of the modified analytic solutions in equation (6) and (7) at different parameters we have conclude that changing in the analytic solution has no effect in the result. When the rate of developing a complication ρ tends to infinity, then the value of C(t) i.e. the number of diabetics with complications and the value of N(t) i.e. the number of diabetics with and without complications will be the same. If the rate of developing a complication is very low then the number of diabetics without complications D(t) is zero. Hence, there is a very high chances to control the complications. When the value of γ i.e.the rate at which complications are controlled is zero then the rate of developing a complication (ρ) is very high. Hence, we have to make effort to decrease the value of I that is the rate of developing incidence of diabetes.

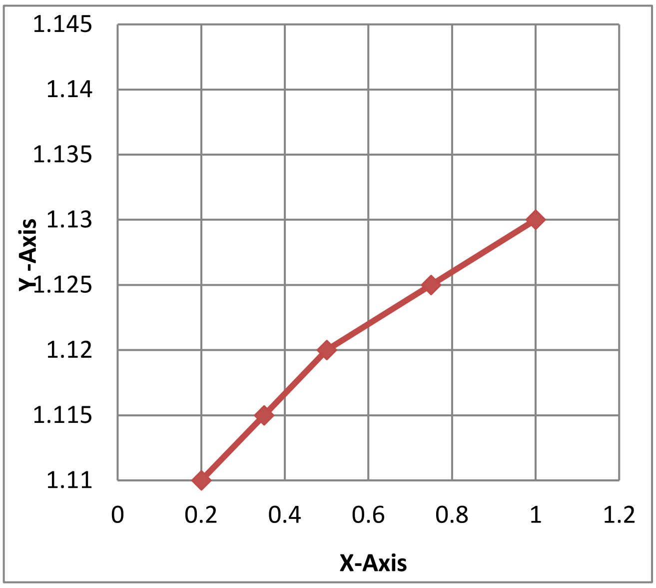

Figure 1:Rate of developing complications based on total diabetic population

In Above Graph:

X-Axis – Rate of Developing Complications(p)

Y-Axis - Number of Diabetic with Complications and Total Number of Diabetic Population.

Scale – 1 Unit = 10^8.

List of parameters used in diabetology and their indicated values are as follows[6]:

N(t): Number of diabetics with and without complications;

C(t): Number of diabetics with complications;

D(t): Number of diabetics without complications;

ρ : rate of developing a complication; $\rho = 0.85$

δ : rate of mortality due to complications; $\delta = 0.05$

ν : rate at which patients with complications become severely disabled; $v = 0.05$

γ : rate at which complications are controlled; $\gamma = 0.50$

μ : natural mortality rate; $\mu = 0.02$

I : Incidence of Diabetes mellitus; $I = 6 \times 10^6$

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