



# $\Delta_{kl}$ - Statistical Convergence via Neutrosophic Normed Spaces for Double Sequences

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ARTICLE INFO	ABSTRACT
<p><b>Published Online:</b> 01 March 2025</p> <p>Corresponding Author: <b>B. G. Ahmadu</b></p>	<p>In this paper, we present the extension of <math>\Delta</math>-statistically convergent and <math>\Delta</math>-statistically Cauchy sequences via neutrosophic normed space (NNS) to double sequences. The study in analogy also define and introduce <math>\Delta_{kl}</math> for which <math>st_{\Delta_{kl}} - \lim_{s_{kl}} = s</math> or <math>s_{kl} - s(S_{kl}^s)</math>, where <math>k \rightarrow \infty, S_{kl}^s</math> denote set of all <math>\Delta_{kl}</math>- statistically convergent sequences. Furthermore, we present their feature utilizing double density and establish some inclusion relations between these concepts and prove some essentials analogous properties for double sequences.</p>
<p><b>KEYWORDS:</b> Double sequences, Statistical sequence, Difference sequence, Neutrosophic normed space</p>	

## 1. INTRODUCTION

The first world publication related to the concept of neutrosophy of the usual notion of convergence. The idea of statistical convergence was given in the first edition (published in Warsaw in 1935) of the monograph of Zygmund (1979), “who called it almost convergence”. Formally the concept of statistical convergence was introduced by Fast (1951) and Steinhaus (1951) independently and later reintroduced by Schoenberg (1959). Although statistical convergence was introduced over nearly last sixty years, it has become an active area of research in recent years. The concept of fuzzy set was originally introduced by Zadeh (1965). The fuzzy theory has become an area of active research for the last fifty years. It has a wide range of applications in the field of science and engineering, population dynamics (2000), chaos control (2004), computer programming (1980), nonlinear dynamical systems (2006), fuzzy physics (1992) and more. The intuitionistic fuzzy set on a universe  $X$  was first introduced by Atansov in (1986) as a generalization of Fuzzy set, where besides the degree of membership of each element to a set, there was considered a degree non – membership. Taking into account the concept of fuzzy set and intuitionistic fuzzy set, Smarandache (2005) introduced the notion of Neutrosophic set (NS) which is a new version of the idea of the classical set. The first world publication related to the concept of neutrosophy was published in (1998) and included in the literature. On the other hand, Kaleva and Seikkala (1984) defined the fuzzy metric spaces (FMS) as a distance between two points to be a non-negative fuzzy number. After that, in (1994) some basic

properties of FMS were studied and the Baire Category. Consequently, FMS has been used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision making and more. After definition of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park (2004) introduced IF metric space (IFMS), that is a generalization of FMS. Then, Park used George and Veeramani’s (1994) work for applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic properties. Moreover, Bera and Mahapatra introduced the neutrosophic soft linear spaces (NSLS) (2017). Later, neutrosophic soft normed linear spaces (NSNLS) was defined by Bera and Mahapatra (2018). Besides, In (2018), neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were defined and studied. Vakeel A. Khan (2021) used the notion of  $\lambda$ -statistical convergence in order to generalize these concepts. And also established some inclusion relations between them. They defined the statistical convergence and  $\lambda$ -statistical convergence in neutrosophic normed space. They gave the  $\lambda$ -statistically Cauchy sequence in neutrosophic normed space and presented the  $\lambda$ -statistically completeness in connection with a neutrosophic normed space. Some interesting examples are also displayed in support of the definitions and results. Kirisci Recently, Kirisci and Simsek (2020), introduced and studied the notion of statistical convergence in a neutrosophic normed spaces. Besides, they showed some interesting results. Recently, Carlos Granados and Alok Dhital (2021) studied and presented the idea of in

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Neutrosophic Normed Spaces. Quite recently, Nazmiye G. B. (2022) studied different types of convergence concepts and applied them to difference sequences. The concept of difference sequences was combined with structures that are advantageous to work like Lacunary sequences. In this work, we shall extend the notion of some concepts of statistical convergence via neutrosophic normed spaces by using double sequences. Moreover, we prove some of its properties and characterizations. Also this work aims to provide a solid foundation for studying statistical convergence properties of double sequences in a neutrosophic normed spaces setting.

2. PRELIMINARIES

**Definition 2.1 (Fast and Steinhaus, 1951):** A number sequence  $x=(x_k)$  is said to be statistically convergent to the number  $L$  if for each  $\epsilon > 0$ ,  $\lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : |x_k - L| \geq \epsilon\}| = 0$

**Definition 2.2 (Fast and Steinhaus, 1951):** A sequence  $x=(x_k)$  is said to be statistically Cauchy sequence if for every  $\epsilon > 0$ , there exists a number  $N = N(\epsilon)$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_{p_k}| \geq \epsilon\}| = 0$$

**Definition 2.3 (Fridy and Khan, 1998):** A sequence  $(x^k)$  is said to be statistically bounded in  $X$  if there exist  $x \in X$  and  $M > 0$  such that  $\Delta(\{k \in \mathbb{N} : \rho(x_k, x) \leq M\}) = 1$ .

**Definition 2.4 (Fridy and Khan, 1998):** A sequence  $(x^k)$  is a statistically Cauchy sequence in  $X$  if for every  $\epsilon > 0$  there exists  $N = N(\epsilon) \in \mathbb{N}$  such that  $\Delta(A_N) \approx 1$ , where

$$A_N(\epsilon) = \{k \in \mathbb{N} : \rho(x_k, x_j) < \epsilon\}.$$

2.2 Statistical Convergence of Double Sequences

The notion of statistical convergence of double sequences was defined by Mursaleen and Edely (2003).

Let  $K \subset \mathbb{N} \times \mathbb{N}$  be two-dimensional set of positive integers and let be  $K(n, m)$  the numbers of  $(j, k)$  in  $K$  such that  $j \leq n$  and  $k \leq m$ . Then, the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of the set  $K \subset \mathbb{N} \times \mathbb{N}$  is defined as:

$$\delta_2(K) = \liminf_{n, m} \frac{K(n, m)}{nm}. \quad \text{Similarly,}$$

The upper asymptotic density of the set  $K \subset \mathbb{N} \times \mathbb{N}$  is defined as:

$$\bar{\delta}_2(K) = \limsup_{n, m} \frac{K(n, m)}{nm}.$$

In case that the sequence  $(\frac{K(n, m)}{nm})$  has a limit in Pringsheim's sense then we say that  $K$  has a double density and is defined as:

$$\lim_{n, m} \frac{K(n, m)}{nm} = \delta(K)$$

Statistical convergence for double sequence  $x=(x_{kj})$  of real which was defined by Mursaleen and Edely (2003) as:

**Definition 2.2.1 (Mursaleen and Edely, 2003):** A real double sequence  $x=(x_{kj})$  is said to be statistically convergent to the number  $L$  if for each  $\epsilon > 0$ , the set

Now, we write the essential definitions and concepts in the study

The concept of statistical convergence was defined by Fast and Steinhaus (1951) and also independently by Buck (1953) and Schornberg (1959) for real and complex sequences.

Further this concept was studied by Salat (1980), Fridy (1985), Connor (1988) and later this notion was studied by various authors. Let  $X$  be a subset of  $\mathbb{N}$

density of  $K$ , denoted by  $d(K)$  is defined as follows:

$$d(K) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

$\{(j, k), j \leq m, k \leq n : |x_{kj} - L| \geq \epsilon\}$ , has a double natural density zero. In this case, we write  $S_2\text{-}\lim x_{kj} = L$ .

**Definition 2.2.2 (Mursaleen and Edely, 2003):** A real double sequence  $x=(x_{jk})$  is said to be statistically Cauchy if for each  $\epsilon > 0$ , there exist positive integers  $m(\epsilon)$  and  $n(\epsilon)$  such that for every  $j, p \geq m$  and  $k, q \geq n$ , the set  $\{(j, k), j \leq m, k \leq n : |x_{jk} - x_{pq}| \geq \epsilon\}$  has double natural density zero.

**Definition 2.2.3 (Mursaleen and Edely, 2003):** A double sequence  $x=(x_{jk})$  is said to be bounded if there exists a real number  $M > 0$  such that  $|x_{jk}| < M$  for each  $j$  and  $k$ , i.e, if  $\|x\|_{(\infty, 2)} = \sup_{j, k} |x_{jk}| < \infty$ . We shall denote the set of all

bounded double sequence by  $l^\infty_2$ .

Note that in contrast to the case of sequence a convergent double sequence need not be bounded.

Triangular norms (t-norms) (TN) were initiated by Menger K (1942). In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t- conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

**Definition 2.2.4 (Kirisci and Simsek, 2020):** Give an

operation  $\circ : [0, 1] \times [0, 1] \rightarrow [0, 1]$ . If the operation  $\circ$  is satisfying the following conditions:

- (1)  $s \circ 1 = s$ ,
- (2) If  $s \leq u$  and  $t \leq v$ , then  $s \circ t \leq u \circ v$ ,
- (3)  $\circ$  is continuous,
- (4)  $\circ$  is continuous and associative.

Then, it is called that the operation  $\circ$  is continuous TN, for  $s, t, u, v \in [0, 1]$ .

**Definition 2.2.5 (Kirisci and Simsek, 2020):** Give an

operation  $\bullet : [0, 1] \times [0, 1] \rightarrow [0, 1]$ . If the operation  $\bullet$  is satisfying the following conditions:

- (1)  $s \bullet 0 = s$ ,
- (2) If  $s \leq u$  and  $t \leq v$ , then  $s \bullet t \leq u \bullet v$ ,
- (3)  $\bullet$  is continuous,
- (4)  $\bullet$  is continuous and associative.

Then, it is called that the operation • is continuous TC, for  $s, t, u, v \in [0, 1]$ .

**Remark:** From the above definitions, we can see that if we take  $0 < \epsilon_1, \epsilon_2 < 1$  for  $\epsilon_1 < \epsilon_2$ , then there exist  $0 < \epsilon_3, \epsilon_4 < 0, 1$  such that  $\epsilon_1 \circ \epsilon_3 \geq \epsilon_2, \epsilon_1 \geq \epsilon_4 \bullet \epsilon_2$ . Moreover, if we take  $\epsilon_5 \in (0, 1)$ , then there exist  $\epsilon_6, \epsilon_7 \in (0, 1)$  such that  $\epsilon_6 \circ \epsilon_6 \geq \epsilon_5$  and  $\epsilon_7 \bullet \epsilon_7 \leq \epsilon_2$ .

The notion of neutrosophic normed space (NNS) was defined by (Kirisci and Simsek, 2020), as well as the definition of statistical convergence with respect to NNS was given.

**Definition 2.2.6 (Kirisci and Simsek, 2020):** Take  $F$  as a vector space  $N = \{ \langle u, G(u), B(u), Y(u) \rangle : u \in F \}$  be a normed space (NS) such that  $N : F \times F^+ \rightarrow [0, 1]$ . Let and show the continuous TN and continuous TC, respectively. If the following conditions are satisfied, then the four-tuple  $\mathcal{V} = (F, N, \circ)$  is called neutrosophic normed space (NNS), for all  $u, v \in F, \lambda, \mu > 0$  and for each  $\theta \neq 0$ :

- (1)  $0 \leq G(u, \lambda) \leq 1, 0 \leq B(u, \lambda) \leq 1, 0 \leq Y(u, \lambda) \leq 1$  for all  $\lambda \in R^+$ ,
- (2)  $G(u, \lambda) + B(u, \lambda) + Y(u, \lambda) \leq 3$ , for all  $\lambda \in R^+$ ,
- (3)  $G(u, \lambda) = 1, \lambda \neq 0$  if and only if  $u = 0$ ,
- (4)  $G(\sigma u, \lambda) = G(u, \frac{\lambda}{|\sigma|})$ ,
- (5)  $G(u, \lambda) \circ G(v, \mu) \leq G(u + v, \lambda + \mu)$ ,
- (6)  $G(u, \cdot)$  is continuous non decreasing function,
- (7)  $\lim_{\lambda \rightarrow \infty} G(u, \lambda) = 1$ ,
- (8)  $B(u, \lambda) = 0, \lambda \neq 0$  if and only if  $u = 0$ ,
- (9)  $B(\sigma u, \lambda) = B(u, \frac{\lambda}{|\sigma|})$ ,
- (10)  $B(u, \mu) \bullet B(v, \lambda) \geq B(u + v, \lambda + \mu)$ ,
- (11)  $B(u, \cdot)$  is continuous non decreasing function,
- (12)  $\lim_{\lambda \rightarrow \infty} B(u, \lambda) = 0$ ,
- (13)  $Y(u, \lambda) = 0, \lambda \neq 0$  if and only if  $u = 0$ ,
- (14)  $Y(\sigma u, \lambda) = Y(u, \frac{\lambda}{|\sigma|})$ ,
- (15)  $Y(u, \lambda) \bullet Y(v, \lambda) \geq Y(u + v, \lambda + \mu)$ ,
- (16)  $Y(u, \cdot)$  is continuous non decreasing function,
- (17)  $\lim_{\lambda \rightarrow \infty} Y(u, \lambda) = 0$ ,
- (18) if  $\lambda \leq 0$ , then  $G(u, \lambda) = 0, B(u, \lambda) = 1$  and  $Y(u, \lambda) = 1$ .

Then  $N = (G, B, Y)$  is called neutrosophic norm (NN).

**Example 2.2.1 (Kirisci and Simsek, 2020):** Let  $(F, \|\cdot\|)$  be a NS. Give the operations  $\circ$  and  $\bullet$  as TN  $u \circ v = u \oplus v$  and  $u \bullet v = u + v - u \oplus v > \|u\|$ ,

$$G(u, \lambda) = \frac{\lambda}{\lambda + \|u\|}, B(u, \lambda) = \frac{\|u\|}{\lambda}, Y(u, \lambda) = \frac{\|u\|}{\lambda}$$

for all  $u, v \in F$ , and  $\lambda > 0$ . If we take  $\lambda \leq \|u\|$ , then  $G(u, \lambda) = 0, B(u, \lambda) = 1$  and  $Y(u, \lambda) = 1$ . Then,  $(F, N, \circ, \bullet)$  is NNS such that  $N : F \times F^+ \rightarrow [0, 1]$ .

**Definition 2.2.7 (Kirisci and Simsek, 2020):** Let  $V$  be a NNS and  $(a_n)$  be a sequence in  $V$  such that  $0 < \epsilon < 1$  and  $\lambda > 0$ . Then,  $(a_n)$  converges to  $a$  if and only if there exists

$n_0 \in \mathbb{N}$  such that  $G(a_n - a, \lambda) > 1 - \epsilon, B(a_n - a, \lambda) < \epsilon$  and  $Y(a_n - a, \lambda) < \epsilon$ . That is  $\lim_{n \rightarrow \infty} G(a_n - a, \lambda) = 1, \lim_{n \rightarrow \infty} B(a_n - a, \lambda) = 0$  and  $\lim_{n \rightarrow \infty} Y(a_n - a, \lambda) = 0$ . In this case, the sequence  $(a_n)$  is said to be a convergent sequence in  $V$ . The convergent in NNS is denoted by  $N\text{-}\lim a_n = L$ .

### 2.3 Δ- Statistical Convergence and Δ- Lacunary Statistical Convergence

**Definition 2.3.1 (Basarir, 1995):**  $(s_k)$  is called to be Δ- statistically convergent to  $s$ , where

$$\delta(\{k \in \mathbb{N} : |\Delta s_k - s| \geq \epsilon\}) = 0, \text{ for all } \epsilon > 0 \text{ and } \Delta s_k = s_k - s_{k+1}, 1.e.,$$

$$\lim_{p \rightarrow \infty} \frac{1}{p} |\{k \leq p : |\Delta s_k - s| \geq \epsilon\}| = 0. \text{ Then, it is demonstrated}$$

$st\text{-}\lim \Delta s_k = s$ .  $S_\Delta$  is denoted, set of all Δ- statistical convergence sequences.

**Definition 2.3.2 (Fridy and Orhan, 1993):** Let  $\theta = \{k_r\}$  be a sequence of increasing integers,  $k_0 = 0$  and also  $\lim_{r \rightarrow \infty} k_r - k_{r-1} = \infty$ . Then,  $\theta$  is called to be Lacunary

sequences. Let  $A \subset \mathbb{N}, I_r = (k_{r-1}, k_r)$  and  $I_r = (k_{r-1}, k_r)$ .  $\delta(A) = \lim_{r \rightarrow \infty} \frac{1}{|I_r|} |\{k \in I_r : k \in A\}|$ , is said to be the  $\theta$ - density of A if limit is exhibited.

**Definition 1.6.5.3 (Fridy and Orhan, 1993):** Let  $A_\epsilon = \{k \in I : |s - s| \geq \epsilon\}$ ; for all  $\epsilon > 0$ ,

$$(A_\epsilon) = \lim_{r \rightarrow \infty} \frac{1}{|I_r|} |\{k \in I_r : |s - s| \geq \epsilon\}| = 0, \text{ in this case,}$$

$(s_k)$  is called to be Δ- Lacunary statistical convergent to  $s$ . Then, it is represented as  $st\text{-}\lim s = s$ .  $S^\theta$  is a denoted set of every Lacunary statistical convergence sequences.

**Definition 2.3.3 (Nazmiye, 2022):** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \nu)}, \mathcal{Q}_{(\mathfrak{R}, \nu)}, \zeta_{(\mathfrak{R}, \nu)}, \otimes, \square)$  be neutrosophic normed spaces and  $\Delta_s = s - s_k$ .  $(s)$  is called to be Δ-

convergence to  $s$  according to neutrosophic normed if, for all  $\epsilon \in (0, 1)$  and  $r > 0$ , there exists a  $k \in \mathbb{N}$  such that, for every  $k \geq \tilde{k}$ ,

$$\mu(\mathfrak{R}, \nu)(\Delta_s - s, r) \leq 1 - \epsilon, \mathcal{Q}_{(\mathfrak{R}, \nu)}((\Delta_s - s, r)) \geq \epsilon, \zeta_{(\mathfrak{R}, \nu)}((\Delta_s - s, r)) \geq \epsilon.$$

This sequences is shown with  $\mathfrak{R}_\Delta\text{-}\lim s_k = s$ .

**Definition 2.3.4 (Nazmiye, 2022):** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \nu)}, \mathcal{Q}_{(\mathfrak{R}, \nu)}, \zeta_{(\mathfrak{R}, \nu)}, \otimes, \square)$  be neutrosophic normed spaces. If there exists  $r > 0$  and  $0 < \epsilon < 1$ , for all  $\Delta_s$  where

$$\mu(\mathfrak{R}, \nu)(\Delta_s, r) \leq 1 - \epsilon, \mathcal{Q}_{(\mathfrak{R}, \nu)}((\Delta_s, r)) \geq \epsilon, \zeta_{(\mathfrak{R}, \nu)}((\Delta_s, r)) \geq \epsilon, \text{ then } (s) \text{ is called } \Delta\text{-bounded sequences in } (\mathcal{U}, \mu_{(\mathfrak{R}, \nu)}, \mathcal{Q}_{(\mathfrak{R}, \nu)}, \zeta_{(\mathfrak{R}, \nu)}, \otimes, \square).$$

**Definition 2.3.5 (Nazmiye, 2022):** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \nu)}, \mathcal{Q}_{(\mathfrak{R}, \nu)}, \zeta_{(\mathfrak{R}, \nu)}, \otimes, \square)$  be neutrosophic normed spaces,  $(s_k)$  is called to be Δ- Cauchy sequence if, for every  $\epsilon \in (0, 1)$  and  $r > 0$ , there exists a  $k_0 \in \mathbb{N}$  such that, for

$$\text{every } k, p \geq k_0, \mu_{(\mathfrak{R}, \nu)}(\Delta_s - \Delta_{s_p}, r) \leq 1 - \epsilon$$

3. MAIN RESULTS

Now, we examine the Δ- Statistical Convergence of Double sequences via Neutrosophic Normed Spaces

3.1 Δ<sub>kl</sub>- Statistical Convergence via Neutrosophic Normed Spaces

**Definition 3.1** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces;  $(s_{kl})$  is called to be Δ<sub>kl</sub> - statistical convergence with respect to  $(\mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})})$  if, for every  $\varepsilon \in (0, 1)$  and  $r > 0$ , there exists  $s$  such that  $\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) \leq 1 - \varepsilon$  or  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \varepsilon\}$ , has natural density zero, i.e.,

$$\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) \leq 1 - \varepsilon \text{ or } \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \varepsilon\}| = 0.$$

Therefore, it will be denoted as  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{kl} s_{kl} = s$  or  $s_{kl} \in S(\mathfrak{R}_{\Delta_{kl}}^{\mathfrak{R}})$ , where  $k \rightarrow \infty$ .  $S(\mathfrak{R}_{\Delta_{kl}}^{\mathfrak{R}})$  denote set of all Δ<sub>kl</sub>- statistical convergence sequences.

**Lemma 3.1** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces and  $(s_{kl})$  is Δ<sub>kl</sub>- statistically convergent in this case,  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{kl} s_{kl}$  is unique.

**Definition 3.2** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces,  $(s_{kl})$  is called to be Δ<sub>kl</sub> - statistical Cauchy sequences if, for every  $\varepsilon \in (0, 1)$  and  $r > 0$ , there exists a  $j, p \in \mathbb{N}$  such that,  $\delta_{(nm)}(\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - \Delta s_{jp}, r) \leq 1 - \varepsilon$  or  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - \Delta s_{jp}, r) \geq \varepsilon\}) = 0$

**Lemma 3.2** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be a Δ<sub>kl</sub>- statistical convergence sequences. Then, for each  $\varepsilon > 0, r > 0$ , the next properties are equivalent:

- (i)  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{kl} s_{kl} = s$
- (ii)  $\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) > 1 - \varepsilon$  or  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \varepsilon\}| = 1$ .
- (iii)  $\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) \leq 1 - \varepsilon\}| = 0$ , and  $\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \varepsilon\}| = 0$ , and  $\lim_{n \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \varepsilon\}| = 0$ ,
- (iv)  $\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) > 1 - \varepsilon\}| = 1$ , and  $\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) < \varepsilon\}| = 1$ , and

$$\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{k \leq n, l \leq m : \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) < \varepsilon\}| = 1$$

(v)  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{n, m \rightarrow \infty} \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) = 1$ , and  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{n, m \rightarrow \infty} \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) = 0$ .

**Lemma 3.3** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces. Every Δ<sub>kl</sub>- statistical convergence sequences is Δ<sub>kl</sub>- statistical Cauchy sequences.

**Proof:** let  $(s_{kl})$  be a Δ<sub>kl</sub>- statistical convergence sequences in  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  and  $st_{\Delta_{kl}}^{\mathfrak{R}} - \lim_{kl} s_{kl} = s$ . For a given  $\varepsilon \in (0, 1)$ , choose  $\vartheta > 0$  such that  $(1 - \varepsilon) \otimes (1 - \varepsilon) > (1 - \mu)$  and  $\varepsilon \square \varepsilon < \vartheta$ . For  $r > 0$ ,

$$\delta_{(n, m)}(G_{n, m}) := \{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - s, r) \leq 1 - \frac{1}{2} \text{ or } \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s, r) \geq \frac{1}{2}\} = 0.$$

Can be written, so  $\delta_{(n, m)}(G_{n, m}) = 1$ . For  $j, p \in G_{n, m}^c$ ,  $\mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{jp} - s, r) > 1 - \varepsilon$  and  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{jp} - s, r) < \varepsilon$ .

Let  $H(n, m) = \{k \leq n, l \leq m : \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{kl} - \Delta s_{jp}, r) \leq 1 - \vartheta$  or  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - \Delta s_{jp}, r) \geq \vartheta\}$ .

It is necessary to show that  $H(n, m) \subset G_{(n, m)}$ . So, to show this, let  $u, w \in (H \cap G^c)$ . In this case,  $\mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{uw} - \Delta s_{jp}, r) \leq 1 - \vartheta$  and  $\mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{uw} - s, r) > 1 - \vartheta$ , especially  $\mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{jp} - s, r) > 1 - \vartheta$ . So,  $1 - \vartheta \geq \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{uw} - \Delta s_{jp}, r) \geq \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}(\Delta s_{uw} - s, r) > 1 - \vartheta$ .

However, this is not possible. Moreover,  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{uw} - \Delta s_{jp}, r) \geq \vartheta$  and  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{uw} - s, r) < \vartheta$ , especially  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{jp} - s, r) < \vartheta$ . Hence,  $\vartheta \leq \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{uw} - \Delta s_{jp}, r) \leq \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{uw} - s, r) < \vartheta$ , which is impossible. With a similar technique, we can apply for  $\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - \Delta s_{jp}, r) \geq \vartheta$ . So,  $H(n, m) \subset G_{(n, m)}$  and  $\delta_{(n, m)}(H) = 0$ . Then, Δ<sub>kl</sub>- statistical Cauchy convergence sequences in  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$ .

**Definition 3.3** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces. If every Δ<sub>kl</sub> - statistical cauchy sequences is Δ<sub>kl</sub> - statistical convergence in  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{V})}^{\mathfrak{Q}}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$ , then this spaces is called complete.

**Theorem 3.3** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$

neutrosophic normed spaces. Then, every  $\Delta_{kl}$ - statistical Cauchy sequences is  $\Delta_{kl}$ - statistical convergence in this spaces.

**Proof:** Let  $(s_{kl})$  be  $\Delta_{kl}$ - statistical Cauchy but not  $\Delta$ - statistical convergent on  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$ . For a

given  $\varepsilon \in (0, 1)$ , choose  $\vartheta > 0$  such that  $(1 - \varepsilon) \otimes (1 - \varepsilon) > (1 - \mu)$  and  $\varepsilon \square \varepsilon < \vartheta$ . Then,

$$\mu_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - \Delta s_{jp}, r) \geq \mu_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - s_j) \otimes \mu_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - s_j) > (1 - \varepsilon) \otimes (1 - \varepsilon) > 1 - \vartheta.$$

$$\varrho_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - \Delta s_{jp}, r) \leq \varrho_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - s_j) \square \varrho_{(\mathfrak{R}, \mathcal{U})}(\Delta s_{kl} - s_j) < \varepsilon \square \varepsilon < \vartheta,$$

$$\zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - \Delta s_{jp}, r) \leq \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s_j) \square \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_{kl} - s_j) < \varepsilon \square \varepsilon < \vartheta.$$

So, for

$$I(n, m) = \{k \leq n, l \leq m: \mu_{(\mathfrak{R}, \mathcal{U})}(\Delta s_k - \Delta s_p, r) \leq 1 - \vartheta \text{ or } \varrho_{(\mathfrak{R}, \mathcal{U})}(\Delta s_k - \Delta s_p, r) \geq \vartheta, \zeta_{(\mathfrak{R}, \mathcal{F})}(\Delta s_k - \Delta s_p, r) \geq \vartheta\},$$

and  $\delta_{(n,m)}(I_{n,m}) = 0$ ; also,  $\delta_{(n,m)}(I_{n,m}) = 1$ . Since  $(s^k)$  is  $\Delta$ - statistical Cauchy, this is impossible. Thus,  $(s^k)$  is  $\Delta$ - statistical convergent in  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$ .

**Result 1.** If  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  is neutrosophic normed spaces, then this spaces is complete.

**Result 2.** Let  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  be neutrosophic normed spaces and  $(s_{kl})$  be a double sequences in this spaces. Then,  $(s_{kl})$  is a  $\Delta_{kl}$ - statistical convergence sequences,  $(s_{kl})$  is a  $\Delta_{kl}$ - statistical Cauchy sequences, and  $(\mathcal{U}, \mu_{(\mathfrak{R}, \mathcal{U})}, \varrho_{(\mathfrak{R}, \mathcal{U})}, \zeta_{(\mathfrak{R}, \mathcal{F})}, \otimes, \square)$  is complete neutrosophic normed spaces.

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