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Special Value of the Odd Zeta Function $\zeta(3)$

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ARTICLE INFO	ABSTRACT
Published Online: 09 December 2024 Corresponding Author: **Takaaki Musha**	Euler clarified even zeta special values which can be written as (rational number)×power of π, but the odd zeta value is unknown. In this paper, the author will try to clarify the special value of $\zeta(3)$ with the aid of the Mathematical program.

KEYWORDS: Riemann zeta function, odd zeta special value, Mathematica, Euler, mathematical program

I. INTRODUCTION

Euler obtained even special values of Riemann zeta function but odd zeta functions are unknown. Euler only obtained the odd zeta function as [1]

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7}\int_0^{\pi/2} x \log(\sin x) dx \quad , \qquad (1)$$

But the integration term was left and the accurate value by using mathematical functions was unknown. Thus Euler tried to estimate the value of $\boldsymbol{\zeta(3)}$ by the numerical calculation as [2]

$$\zeta(3) = \alpha(\log 2)^3 + \beta \frac{\pi^2}{6} \log 2, \qquad (2)$$

where α and β are rational numbers. But no one could prove this equation.

Mathematica launched by Wolfram Research Inc. can be used not only to solve complex math problem but also to discover new patterns and relationships and gain insight and intuition of the problem. The author tries to clarify the special value of $\boldsymbol{\zeta(3)}$ with the aid of the computer program Mathematica 5.0 [3], and the formula of $\boldsymbol{\zeta(3)}$ without the integration term can be obtained.

II. ESTIMATION OF THE ZETA SPECIAL VALUE BY MATHEMATICA

The equation of (1) can be modified by using partial integration formula as

$$\int_0^{\pi/2} x \log(\sin x) dx = -\int_0^{\pi/2} \frac{x^2}{2} \cot x \, dx \quad , \qquad (3)$$

From which, we consider the integration $\int x^2 \cot x \, dx$, we have the equation by using Mathematica shown as

$$\int_a^{\pi/4} x^2 \cot x \, dx = -\frac{i}{3}a^3 + \frac{Catalan}{4}\pi + \frac{\pi^2}{32}\log 2 - a^2 \log(1 - e^{-2ia}) - iaPolyLog[2, e^{-2ia}] - \frac{1}{2}PolyLog[3, e^{-2ia}] - \frac{3}{64}\zeta(3) \qquad (4)$$

To eliminate the integration term by

$$\lim_{a\to\pi/4}\int_a^{\pi/4} x^2 \cot x \, dx = -i\frac{\pi^3}{192} + \frac{Catalan}{4}\pi + \frac{\pi^2}{32}\log 2 - \frac{\pi^2}{16}\log(1+i) - \frac{\pi}{4}(Catalan - i\frac{\pi^2}{48}) - \frac{1}{2}PolyLog[3, -i] - \frac{3}{64}\zeta(3) = 0, \qquad (5)$$

Taking real part of this equation, we have

$$\frac{3}{64}\zeta(3) = \frac{\pi^2}{32}\log 2 - \frac{\pi^2}{16}Re[\log(1+i)] - \frac{1}{2}Re[PolyLog[3, -i]], \qquad (6)$$

where Re is the real part of the number.

Rearranging them, we finally obtain

$$\zeta(3) = \frac{2\pi^2}{3}\log 2 - \frac{4}{3}\pi^2 Re[\log(1+i)] - \frac{32}{3}Re[Li_3(-i)] \, , \qquad (7)$$

where $Li_3(-i) = PolyLog[3, -i]$,(polylogarithm)

According to the calculation by Mathematica, $\frac{4\pi^2}{3}Im[\log(1+i)] + \frac{32}{3}Im[Li_3(-i)] = 0$, then we have

$$\zeta(3) = \frac{2\pi^2}{3}\log 2 - \frac{4\pi^2}{3}\log(1+i) - \frac{32}{3}Li_3(-i), \qquad (8)$$

This formula can be also obtain from the following integration.

III. DERIVATION OF THE FORMULA OF $\boldsymbol{\zeta(3)}$ FROM ANOTHER INTEGRATION

We can obtain the same formula from

$$\int_a^{\pi/4} x^3 \operatorname{cosec}^2 x \, dx = -ia^3 + \frac{3Catalan}{4}\pi - \frac{\pi^3}{64} + a^3 \cot a + \frac{1}{64}\pi^2 \log[64] - 3a^2 \log(1 - e^{-2ia}) - 3iaPolyLog[2, e^{-2ia}] - \frac{3}{2}PolyLog[3, e^{-2ia}] - \frac{9}{64}\zeta(3), \qquad (9)$$

Eliminating the integration term and taking the real part of the equation, we have

$$\mathrm{Re}\left[\lim_{a\to\pi/4}\int_a^{\pi/4} x^3 \operatorname{cosec}^2 x dx\right]$$
$$= \frac{3Catalan}{4}\pi + \frac{\pi^2}{64}log[64] - 3\left(\frac{\pi}{4}\right)^2 Re[log(1 + i)] - \frac{3}{4}Catalan \cdot \pi - \frac{3}{2}Re[PolyLog[3, -i]] - \frac{9}{64}\zeta(3) = 0, \quad (10)$$

Rearranging them, we finally obtain

$$\zeta(3) = \frac{2\pi^2}{3}log\,2 - \frac{4\pi^2}{3}[log(1 + i)] - \frac{32}{3}[Li_3(-i)], \quad (11)$$

which is identical to Eq,(8).

IV. CONCLUSION

By the mathematical calculation using Mathematica, the formula of the special vale of $\zeta(3)$ can be obtained shown as

$$\zeta(3) = \frac{2\pi^2}{3}log\,2 - \frac{4\pi^2}{3}[log(1 + i)] - \frac{32}{3}[Li_3(-i)] = 1.202056903159594285 39 \cdots \quad (12)$$

which is identical to the numerical value of the special value of $\zeta(3)$ [4].

From which, we can derive the simplest form of the special value of $\zeta(3)$, which could not be obtained by Euler. From this equation, it is considered that odd zeta special values are very different from even zeta special values.

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