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Multidimensional Generalized Geometric Progressions of Multiplicity One

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ABSTRACT

This paper is a review article to expose the extended advanced concepts of geometric progressions made by applying the basic concepts of an arbitrary dimension and a fixed multiplicity (one which can be expanded for more than one also) applied on different common ratios, which was earlier published in chapter seven and eight in two books on multidimensional geometric progressions cited in the references. In this article we will report the advanced geometric progressions with multiplicity one of different dimensions from one to r and will discuss the formulae to find their general terms and the sums of first n terms. The formulae for infinite number of terms for different dimensions and multiplicities have also been discussed. We have left the discussion on the formulae to find the geometric means between any two arbitrary terms of such generalized geometric progressions so that mathematics teachers and learners can find it useful in teaching with a new look and a research-oriented approach. The article also opens many new areas of research and its applications.

KEYWORDS: geometric progression, generalized geometric progression, common ratios, dimension, multiplicity, sum of finite terms, sum of infinite terms, etc.

1. INTRODUCTION

The study of patterns contributes a lot in mathematical generalizations. A succession of numbers (either real or complex numbers) called as the first term, the second term, the third term and so on gives rise to a sequence i.e., a progression. The progression is divided into arithmetic, geometric, harmonic, geometric-arithmetic, etc. By combining any two we get another type of progression. For example: a combination of an arithmetic progression and a geometric progression produce an arithmetic-geometric series (James & James, 2001; Katz, 2019; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

A progression is a set of numbers (may be repeated or different) written in a fixed order followed by a predefined rule. The terms of the progressions are known as elements or terms or members. In a progression we can repeat it multiple times in a definite order. In progression the order of the terms matter. It is finite or infinite according as the number of terms. It is denoted by its nth term, where n states the nth element of the progression. We can represent the finite progression having its nth term as t_n by $(t_n)_{n=1}^{r}$, where r is a natural number. The infinite sequence for the same nth term of the sequence is denoted by $(t_n)_{n=1}^{\infty}$, where n is a natural number (James & James, 2001; Progression - Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

A succession of numbers is said to be in geometric progression if the ratio of any term and its preceding term is constant throughout. The constant is called the common ratio of the geometric progression. The first term in the progression from where it starts is called the first term or initial term. The sum of the members of a progression gives a series denoted by $\sum_{n=1}^{\infty} t_n = t_1 + t_2 + t_3 + \cdots$, where t_n is the nth term of the progression. The partial sum of the series is obtained by replacing the infinity symbol with a finite number (say N) as $\sum_{n=1}^{N} t_n = t_1 + t_2 + t_3 + \cdots + t_N$. Generally it is denoted by S_N. If the partial sum converges, we say that the infinite series is convergent (Convergent series - Wikipedia contributors, 2025; Geometric progression - Wikipedia contributors, 2026; Progression -

Wikipedia contributors, 2025; Sequence - Wikipedia contributors, 2026; Series (mathematics) - Wikipedia contributors, 2026; Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

2. PRELIMINARIES

To proceed on the proposed article, we should have the basic knowledge of:

2.1. Geometric Progression: If a be the first term and r is the common ratio of a G. P., then the standard or general form of an G. P. is given by $a, ar, ar^2, ar^3, ar^4, \ldots$. For example, the progression 3, 6, 12, 24, 48, 96, ... is a geometric progression with first term 3 and common ration 2 (Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

2.2. General Term: If the first term is a and the common ratio is r, then the nth term or general term of the geometric progression is given by $t_n = ar^{n-1}$ (Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

2.3. Geometric Series: The sum of all the terms of a geometric progression is called a geometric series. The sum of a geometric series consisting of n terms with 'a' as the first term and 'r' as common ratio is given by $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$, where S_n denotes the sum of first n terms of the series. Sum of the infinite terms of the geometric series for -1 < r < 1 is given by $S_\infty = \frac{a}{1-r}$ (Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

2.4. K-Dimensional Generalized Geometric Progression: A multiple geometric progression or generalized geometric progression or k-dimensional geometric progression or k-dimensional generalized geometric progression is a set of numbers constructed as a geometric progression but allowing k possible common ratios one by one periodically. The number k that is the number of permissible common ratios is called the dimension of the generalized geometric progression. For example, let us start with 3 and multiply 3 and 2, repeatedly we get a 2-dimensional generalized geometric progression as 3, 9, 18, 54, 108, 324, 648, ... (Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

2.5. K-Dimensional Generalized Geometric Progression with Multiplicity m: If in k- dimensional geometric progression, every common ratio is applied m times successively at a time and periodically, then we get a k-D.G.P. with multiplicity m. Here the number of times (m) a common ratio is applied has been named as multiplicity of the k-D.G.P. In the present article, we have denoted it by k-D.G.P.(m) or by k-D.G.P.(m). The example discussed above 3, 9, 18, 54, 108, 324, 648, ... is a 2-dimensional geometric progression of multiplicity one. The geometric progression we study is an example of 1-dimensional geometric progression of multiplicity one. We name it as one-dimensional geometric progression. R-dimensional geometric progression with multiplicity one consists of R common ratio in which each one is applied one after another consecutively and periodically (Yadav, 2008a, 2008b, 2010, 2020a, 2020b, 2026a, 2026b, 2026c; Yadav et al., 2026a, 2026b).

3. DISCUSSION

As far as the previous works on multidimensional generalized progressions are concerned, Yadav (2008a) propounded two-dimensional generalized arithmetic progressions of multiplicity one, Yadav (2008b) propounded two-dimensional generalized arithmetic progressions of multiplicity two, Yadav (2010) propounded three-dimensional generalized arithmetic progressions of multiplicity one. Then Yadav (2020a) extended it for three-dimensional generalized arithmetic progressions of multiplicity two and three and published them in his book 'Multi-Dimensional Arithmetic Progression' with a brief discussion on the multidimensional generalized arithmetic progressions of multiplicity one and the scope of further extension. The same book was reproduced with Indian edition by Yadav (2026a) with the title 'Multidimensional Arithmetic Progressions: Research in Arithmetic Progressions, Volume-I'. Recently Yadav et al. (2026a, 2026b) and Yadav (2026c) published the chapters five, six and seven as review articles respectively.

Thereafter Yadav (2020b) extended the concepts of multidimensional generalized arithmetic progressions for geometric progressions and introduced multidimensional generalized geometric progressions for dimensions one, two, three with multiplicities one, two and three in chapters two to six and then provided a brief discussion on the multidimensional generalized geometric progressions of multiplicity one and its scope of further extension. Yadav (2026b) published the same book with Indian edition with the title 'Multidimensional Geometric Progressions: Research in Geometric Progressions, Volume-I' with a note on the further possible extension for different dimensions and multiplicities. In this review article, the multidimensional generalized geometric progressions of multiplicity one has been reproduced with following sections:

3.1. R-Dimensional Geometric Progression of Multiplicity One: The standard form of R-D.G.P.(1) is

$$a$$
$$ar_1$$
$$ar_1r_2$$
$$ar_1r_2r_3$$
$$\ldots$$
$$ar_1r_2r_3 \ldots r_R$$

$$ar_1^2 r_2 r_3 \dots r_R$$
$$ar_1^2 r_2^2 r_3 \dots r_R$$
$$ar_1^2 r_2^2 r_3^2 \dots r_R^2$$
$$ar_1^3 r_2^2 r_3^2 \dots r_R^2$$
$$\dots$$
$$ar_1^\alpha r_2^\alpha r_3^\alpha \dots r_R^\alpha$$
$$\dots$$

and so on (Yadav, 2020b, 2026b).

3.2. General Term: We will discuss the general formulae for nth terms of different geometric progressions of multiplicity one with different dimensions and then will generalize it for R-dimension geometric progression of multiplicity one. We know that for:

3.2.1. One Dimensional Geometric Progression of Multiplicity One: The standard form is

$$a, ar_1, ar_1^2, ar_1^3, \dots$$

and its general term for nth term is given by

$$T_n = ar_1^{n-1}$$

(Yadav, 2020b, 2026b).

3.2.2. Two-Dimensional Geometric Progression of Multiplicity One: The standard form is

$$a, ar_1, ar_1r_2, ar_1^2r_2, ar_1^2r_2^2, ar_1^3r_2^2, \dots$$

and its general term is given by two different formulae depending on the values of n as:

$$T_{2m-1} = t_n = t_{2m-1} = t_{1,3,5,\dots} = a(r_1r_2)^{m-1} = a(r_1r_2)^{\frac{n-1}{2}}$$

$$T_{2m} = t_n = t_{2m} = t_{2,4,6,\dots} = ar_1(r_1r_2)^{m-1} = ar_1(r_1r_2)^{\frac{n-2}{2}}$$

(Yadav, 2020b, 2026b).

3.2.3. Three-Dimensional Geometric Progression of Multiplicity One: If a be the first term and r_1, r_2, r_3 be three consecutive common ratios, then the general or standard form of three-dimensional geometric progression with multiplicity one is:

$$a$$
$$ar_1$$
$$ar_1r_2$$
$$ar_1r_2r_3$$
$$ar_1^2r_2r_3$$
$$ar_1^2r_2^2r_3$$
$$ar_1^2r_2^2r_3^2$$
$$ar_1^3r_2^2r_3^2$$
$$ar_1^3r_2^3r_3^2$$
$$ar_1^3r_2^3r_3^3$$
$$\dots$$

and so on. For example, 2, 4, 12, 48, 96, 288, 1152, … is a three-dimensional geometric progression with multiplicity one having first term 2 and common ratios 2, 3 and 4. Its general term is given by three different formulae depending on the values of n as:

$$T_{3m-2} = t_n = t_{3m-2} = t_{1,4,7,\dots} = a(r_1r_2r_3)^{m-1}$$
$$= a(r_1r_2r_3)^{\frac{n-1}{3}}$$

$$T_{3m-1} = t_n = t_{3m-1} = t_{2,5,8,\dots} = ar_1(r_1r_2r_3)^{m-1}$$
$$= ar_1(r_1r_2r_3)^{\frac{n-2}{3}}$$

$$T_{3m} = t_n = t_{3m} = t_{3,6,9,\dots} = ar_1r_2(r_1r_2r_3)^{m-1}$$
$$= ar_1r_2(r_1r_2r_3)^{\frac{n-3}{3}}$$

(Yadav, 2020b, 2026b).

3.2.4. Four-Dimensional Geometric Progression of Multiplicity One: The standard form is

$$a$$
$$ar_1$$
$$ar_1r_2$$
$$ar_1r_2r_3$$
$$ar_1r_2r_3r_4$$
$$ar_1^2r_2r_3r_4$$
$$ar_1^2r_2^2r_3r_4$$
$$\dots$$

Following the previous steps discussed earlier its general term can be given by four different formulae depending on the values of n as:

$$T_{4m-3} = t_n = t_{4m-3} = t_{1,5,9,\dots} = a(r_1r_2r_3r_4)^{m-1}$$
$$= a(r_1r_2r_3r_4)^{\frac{n-1}{4}}$$

$$T_{4m-2} = t_n = t_{4m-2} = t_{2,6,10,\dots} = ar_1(r_1r_2r_3r_4)^{m-1}$$
$$= ar_1(r_1r_2r_3r_4)^{\frac{n-2}{4}}$$

$$T_{4m-1} = t_n = t_{4m-1} = t_{3,7,11,\dots} = ar_1r_2(r_1r_2r_3r_4)^{m-1}$$
$$= ar_1r_2(r_1r_2r_3r_4)^{\frac{n-3}{4}}$$

$$T_{4m} = t_n = t_{4m} = t_{4,8,12,\dots} = ar_1r_2r_3(r_1r_2r_3r_4)^{m-1}$$
$$= ar_1r_2r_3(r_1r_2r_3r_4)^{\frac{n-4}{4}}$$

(Yadav, 2020b, 2026b).

Proceeding in the same way we can generalize R-Dimensional Geometric Progression of Multiplicity One**.** The standard form is

$$a$$
$$ar_1$$
$$ar_1r_2$$
$$ar_1r_2r_3$$
$$\dots$$
$$ar_1r_2r_3 \dots r_R$$
$$ar_1^2r_2r_3 \dots r_R$$
$$ar_1^2r_2^2r_3 \dots r_R$$
$$ar_1^2r_2^2r_3^2 \dots r_R^2$$
$$ar_1^3r_2^2r_3^2 \dots r_R^2$$
$$\dots$$
$$ar_1^\alpha r_2^\alpha r_3^\alpha \dots r_R^\alpha$$
$$\dots$$

(Yadav, 2020b, 2026b).

Following the steps discussed earlier its general term can be given by R different formulae as:

When n = 1, 1 + R, 1 + 2R, 1 + 3R, 1 + 4R, … = Rm-(R-1) for m = 1, 2, 3, … terms are taken into consideration, then the general term corresponding to these is given by

$$T_{Rm-(R-1)} = t_n = t_{Rm-(R-1)} = a(r_1r_2r_3 \dots r_R)^{m-1}$$

$$= a(r_1r_2r_3 \dots r_R)^{\frac{n-1}{R}}$$

When n = 2, 2 + R, 2 + 2R, 2 + 3R, 2 + 4R, … = Rm-(R-2) for m = 1, 2, 3, … terms are taken into consideration, then the general term corresponding to these is given by

$$T_{Rm-(R-2)} = t_n = t_{Rm-(R-2)} = ar_1(r_1r_2r_3 \dots r_R)^{m-1}$$
$$= ar_1(r_1r_2r_3 \dots r_R)^{\frac{n-2}{R}}$$

When n = 3, 3 + R, 3 + 2R, 3 + 3R, 3 + 4R, … = Rm-(R-3) for m = 1, 2, 3, … terms are taken into consideration, then the general term corresponding to these is given by

$$T_{Rm-(R-3)} = t_n = t_{Rm-(R-3)} = ar_1r_2(r_1r_2r_3 \dots r_R)^{m-1}$$
$$= ar_1r_2(r_1r_2r_3 \dots r_R)^{\frac{n-3}{R}}$$

Similarly for arbitrary value of $n = \alpha, \alpha + R, \alpha + 2R, \alpha + 3R, \dots = Rm - (R - \alpha)$, we have the general term

$$T_{Rm-(R-\alpha)} = t_n = t_{Rm-(R-\alpha)}$$
$$= ar_1r_2 \dots r_{\alpha-1}(r_1r_2r_3 \dots r_R)^{m-1}$$
$$= ar_1r_2 \dots r_{\alpha-1}(r_1r_2r_3 \dots r_R)^{\frac{n-\alpha}{R}}$$

And in the last, for n = R, 2R, 3R, … = mR, the general term is given by

$$T_{Rm} = t_n = t_{Rm} = ar_1r_2 \dots r_{R-1}(r_1r_2r_3 \dots r_R)^{m-1}$$
$$= ar_1r_2 \dots r_{R-1}(r_1r_2r_3 \dots r_R)^{\frac{n-R}{R}}$$

3.2.5. Rule to Write the General Formula for General Term: First we write the general term of the subscripts of different terms as

$$T_{Rm-(R-\alpha)} = t_n = t_{Rm-(R-\alpha)}$$

Then the general term would be written as

$$ar_1r_2 \dots r_{\alpha-1}(r_1r_2r_3 \dots r_R)^{m-1}$$

where $ar_1r_2 \dots r_{\alpha-1}$ is the product of α number of terms of first term a and first $(\alpha - 1)$ common ratios $r_1, r_2, \dots, r_{\alpha-1}$. That is, the general formula of all the above formulae is

$$T_{Rm-(R-\alpha)} = t_n = t_{Rm-(R-\alpha)}$$
$$= ar_1r_2 \dots r_{\alpha-1}(r_1r_2r_3 \dots r_R)^{m-1}$$
$$= ar_1r_2 \dots r_{\alpha-1}(r_1r_2r_3 \dots r_R)^{\frac{n-\alpha}{R}}$$

3.3. Sum Formulae: We know that for one dimensional geometric progression with multiplicity one, we get the series

$$a + ar_1 + ar_1^2 + ar_1^3 + \cdots$$

and its sum is given by

$$S_n = \frac{a(r_1^n - 1)}{(r_1 - 1)} = \frac{a(1 - r_1^n)}{(1 - r_1)}$$

For $n \to \infty$ and $|r_1| < 1$, we have

$$S_\infty = \frac{a}{(1 - r_1)}$$

Applying these, we find the sum formulae of the following:

3.3.1. For two-dimensional geometric progression with multiplicity one, we get the following series

$$a + ar_1 + ar_1r_2 + ar_1^2r_2 + ar_1^2r_2^2 + ar_1^3r_2^2 + \cdots$$

and its sum is given by two different formulae depending on the values of n as:

(i) When the number of terms is even i.e., n = 2m, for m = 1, 2, 3, …, then

$$S_n = \frac{a\left\{1 - (r_1r_2)^{\frac{n}{2}}\right\}}{(1 - r_1r_2)} + \frac{ar_1\left\{1 - (r_1r_2)^{\frac{n}{2}}\right\}}{(1 - r_1r_2)}$$

(ii) When the number of terms is odd i.e., n = 2m-1, for m = 1, 2, 3, …, then

$$S_n = \frac{a\left\{1 - (r_1r_2)^{\frac{n+1}{2}}\right\}}{(1 - r_1r_2)} + \frac{ar_1\left\{1 - (r_1r_2)^{\frac{n-1}{2}}\right\}}{(1 - r_1r_2)}$$

For $n \to \infty$ and $|r_1| < 1, |r_2| < 1$, we have

$$S_\infty = \frac{a(1 + r_1)}{(1 - r_1r_2)}$$

3.3.2. For three-dimensional geometric progression with multiplicity one, we get the series

$$a + ar_1 + ar_1r_2 + ar_1r_2r_3 + ar_1^2r_2r_3 + ar_1^2r_2^2r_3 + ar_1^2r_2^2r_3^2 + \cdots$$

and its sum is given by three formulae depending on the values of n as:

(i) When the number of terms is of the form n = 1, 4, 7, 10, … = 3m-2, for m = 1, 2, 3, … then

$$S_n = \frac{a\left\{1 - (r_1r_2r_3)^{\frac{n+2}{3}}\right\}}{(1 - r_1r_2r_3)} + \frac{ar_1(1 + r_2)\left\{1 - (r_1r_2r_3)^{\frac{n-1}{3}}\right\}}{(1 - r_1r_2r_3)}$$

(ii) When the number of terms is of the form n = 2, 5, 8, 11, … = 3m-1, for m = 1, 2, 3, …, then

$$S_n = \frac{a(1 + r_1)\left\{1 - (r_1r_2r_3)^{\frac{n+1}{3}}\right\}}{(1 - r_1r_2r_3)} + \frac{ar_1r_2\left\{1 - (r_1r_2r_3)^{\frac{n-2}{3}}\right\}}{(1 - r_1r_2r_3)}$$

(iii) When the number of terms is of the form n = 3, 6, 9, 12, … = 3m, for m = 1, 2, 3, …, then

$$S_n = \frac{a(1 + r_1 + r_1r_2)\left\{1 - (r_1r_2r_3)^{\frac{n}{3}}\right\}}{(1 - r_1r_2r_3)}$$

For $n \to \infty$ and $|r_1| < 1, |r_2| < 1, |r_3| < 1$, we have

$$S_\infty = \frac{a(1 + r_1 + r_1r_2)}{(1 - r_1r_2r_3)}$$

3.3.3. For four-dimensional geometric progression with multiplicity one, we get the series

$$a + ar_1 + ar_1r_2 + ar_1r_2r_3 + ar_1r_2r_3r_4 + ar_1^2r_2r_3r_4 + ar_1^2r_2^2r_3r_4 + ar_1^2r_2^2r_3^2r_4 + \cdots$$

and its sum is given by four formulae depending on the values of n as:

(i) When the number of terms is of the form n = 1, 5, 9, 13, … = 4m-3, for m = 1, 2, 3, …, then

$$S_n = \frac{a\left\{1 - (r_1r_2r_3r_4)^{\frac{n+3}{4}}\right\}}{(1 - r_1r_2r_3r_4)} + \frac{ar_1(1 + r_2 + r_2r_3)\left\{1 - (r_1r_2r_3r_4)^{\frac{n-1}{4}}\right\}}{(1 - r_1r_2r_3r_4)}$$

(ii) When the number of terms is of the form n = 2, 6, 10, 14, … = 4m-2, for m = 1, 2, 3, …, then

$$S_n = \frac{a(1+r_1)\left\{1-(r_1r_2r_3r_4)^{\frac{n+2}{4}}\right\}}{(1-r_1r_2r_3r_4)} + \frac{ar_1r_2(1+r_3)\left\{1-(r_1r_2r_3r_4)^{\frac{n-2}{4}}\right\}}{(1-r_1r_2r_3r_4)}$$

(iii) When the number of terms is of the form n = 3, 7, 11, 15, … = 4m-1, for m = 1, 2, 3, …, then

$$S_n = \frac{a(1+r_1+r_1r_2)\left\{1-(r_1r_2r_3r_4)^{\frac{n+1}{4}}\right\}}{(1-r_1r_2r_3r_4)} + \frac{ar_1r_2r_3\left\{1-(r_1r_2r_3r_4)^{\frac{n-3}{4}}\right\}}{(1-r_1r_2r_3r_4)}$$

(iv) When the number of terms is of the form n = 4, 8, 12, 16, … = 4m, for m = 1, 2, 3, …, then

$$S_n = \frac{a(1+r_1+r_1r_2+r_1r_2r_3)\left\{1-(r_1r_2r_3r_4)^{\frac{n}{4}}\right\}}{(1-r_1r_2r_3r_4)}$$

For $n \to \infty$ and $|r_1| < 1$, $|r_2| < 1$, $|r_3| < 1$, $|r_4| < 1$, we have

$$S_\infty = \frac{a(1+r_1+r_1r_2+r_1r_2r_3)}{(1-r_1r_2r_3r_4)}$$

Proceeding in the way discussed above, we can generalize the formulae for sum of the first n terms for R dimensional geometric progression of multiplicity one. For it, we get the series

$$a + ar_1 + ar_1r_2 + ar_1r_2r_3 + \ldots + ar_1r_2r_3 \ldots r_R + ar_1{}^2r_2r_3 \ldots r_R + \cdots + ar_1{}^2r_2{}^2r_3{}^2 \ldots r_R{}^2 + \cdots + ar_1{}^\alpha r_2{}^\alpha r_3{}^\alpha \ldots r_R{}^\alpha + \cdots$$

Following the steps discussed above its sum for first n terms will be given by R different formulae depending on the values of n as follows:

(i) When the number of terms is n = 1, 1+R, 1+2R, 1+3R, … = Rm-(R-1), then the sum is given by

$$S_n = \frac{a\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n+R-1}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)} + \frac{ar_1(1+r_2+r_2r_3+r_2r_3r_4+\cdots+r_2r_3\ldots r_{R-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n-1}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)}$$

(ii) When the number of terms is n = 2, 2+R, 2+2R, 2+3R, … = Rm-(R-2), then the sum is given by

$$S_n = \frac{a(1+r_1)\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n+R-2}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)} + \frac{ar_1r_2(1+r_3+r_3r_4+r_3r_4r_5+\cdots+r_3r_4\ldots r_{R-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n-2}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)}$$

(iii) When the number of terms is n = 3, 3+R, 3+2R, 3+3R, … = Rm-(R-3), then the sum is given by

$$S_n = \frac{a(1+r_1+r_1r_2)\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n+R-3}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)} + \frac{ar_1r_2r_3(1+r_4+r_4r_5+r_4r_5r_6+\cdots+r_4r_5\ldots r_{R-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n-3}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)}$$

Similarly for n = α, α+R, α+2R, α+3R, … = Rm-(R- α), the sum is given by

$$S_n = \frac{a(1+r_1+r_1r_2+\cdots+r_1r_2\ldots r_{\alpha-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n+R-\alpha}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)} + \frac{ar_1r_2\ldots r_\alpha(1+r_{\alpha+1}+r_{\alpha+1}r_{\alpha+2}+r_{\alpha+1}r_{\alpha+2}r_{\alpha+3}+\cdots+r_{\alpha+1}r_{\alpha+2}\ldots r_{R-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n-\alpha}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)}$$

Similarly for n = R, 2R, 3R, … = Rm = Rm-(R-R), the sum is given by

$$S_n = \frac{a(1+r_1+r_1r_2+r_1r_2r_3+\cdots+r_1r_2\ldots r_{R-1})\left\{1-(r_1r_2r_3\ldots r_R)^{\frac{n}{R}}\right\}}{(1-r_1r_2r_3\ldots r_R)}$$

Also for $n \to \infty$ and $|r_1| < 1$, $|r_2| < 1$, $|r_3| < 1$, $|r_4| < 1$, … , $|r_R| < 1$, we have

$$S_\infty = \frac{a(1+r_1+r_1r_2+r_1r_2r_3+\cdots+r_1r_2\ldots r_{R-1})}{(1-r_1r_2r_3\ldots r_R)}$$

Example 3.1. For 6-D.G.P.(1), where 3 is the first term and 2, 3, 2, 4, 3, 5 are consecutive common ratios, let us write down (a) first 10 terms of the progression, (b) the 12th and 13th terms, and (c) the sum of first 10 terms.

We have

$a = 3, r_1 = 2, r_2 = 3, r_3 = 2, r_4 = 4, r_5 = 3, r_6 = 5$.

Therefore

(a) The 6-D.G.P.(1) is

3, 6, 18, 36, 144, 432, 2160, 4320, 12960, 25920, …

(b) The required terms are

$$t_{12} = t_{6m(m=2)} = ar_1r_2r_3r_4r_5(r_1r_2r_3r_4r_5r_6)^{m-1} = ar_1r_2r_3r_4r_5(r_1r_2r_3r_4r_5r_6)^{2-1} = 3.144(720)^1 = 311040$$

and

$$t_{13} = t_{6m-5(m=3)} = a(r_1r_2r_3r_4r_5r_6)^{m-1} = a(r_1r_2r_3r_4r_5r_6)^{3-1} = 3(720)^2 = 1555200$$

(c) The sum of first 10 terms is

$$S_{10} = S_{6m-2(m=2)} = \frac{a(1+r_1+r_1r_2+r_1r_2r_3)\{(r_1r_2r_3r_4r_5r_6)^2-1\}}{r_1r_2r_3r_4r_5r_6-1} + \frac{ar_1r_2r_3r_4(1+r_5)\{(r_1r_2r_3r_4r_5r_6)^1-1\}}{r_1r_2r_3r_4r_5r_6-1}$$
$$= \frac{3(1+2+6+12)\{(720)^2-1\}}{720-1} + \frac{144(1+3)\{720-1\}}{720-1} = 45423 + 576 = 45999$$

Example 3.2. Find the sum of infinite series of the 6-D.G.P.(1), where

$$a = 3, r_1 = \frac{1}{2}, r_2 = \frac{1}{3}, r_3 = \frac{1}{2}, r_4 = \frac{1}{4}, r_5 = \frac{1}{3}, r_6 = \frac{1}{5}.$$

We have all $|r_i| < 1$, for i = 1, 2, 3, 4, 5, 6. Therefore we have

$$S_\infty = \frac{a(1 + r_1 + r_1r_2 + r_1r_2r_3 + r_1r_2r_3r_4 + r_1r_2r_3r_4r_5)}{(1 - r_1r_2r_3r_4r_5r_6)}$$
$$= \frac{3\left(1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{48} + \frac{1}{144}\right)}{\left(1 - \frac{1}{720}\right)} = \frac{3780}{719}$$

4. CONCLUSION

Two properties of geometric progression of dimension one and multiplicity one has been extended for higher dimension and multiplicities, the general terms and the sum of the n terms. Which is a limitation of this review article because we cannot write the general terms and sum of the series in a single formula. The sum of the infinite terms for increasing dimensions follows the same and similar forms. We have not discussed the geometric means between two terms for higher dimension and multiplicity, although we can write them for different cases.

5. SCOPE OF FUTURE WORK

Multidimensional geometric progression or multidimensional generalized geometric progression of different multiplicities means r-dimensional geometric progression with multiplicity m, where r = 1, 2, 3, 4, … and m = 1, 2, 3, 4, … In the present article only six types of geometric progressions 1-D.G.P.(1), 2-D.G.P.(1), 2-D.G.P.(2), 3-D.G.P.(1), 3-D.G.P.(2), 3-D.G.P.(3), and R-D.G.P.(1) have been discussed, which can be further extended for: 1-dimensional geometric progression with multiplicity 2, 1-dimensional geometric progression with multiplicity 3, …, 1-dimensional geometric progression with multiplicity m, …, 2-dimensional geometric progression with multiplicity 3, 2-dimensional geometric progression with multiplicity 4, … , 2-dimensional geometric progression with multiplicity m, … , 3-dimensional geometric progression with multiplicity 4, 3-dimensional geometric progression with multiplicity 5, … , 3-dimensional geometric progression with multiplicity m, … , R-dimensional geometric progression with multiplicity 2, R-dimensional geometric progression with multiplicity 3, … , R-dimensional geometric progression with multiplicity m, …, etc. Although 1-D.G.P.(m) is nothing special but 1-D.G.P.(1). The concept of multiplicity does not change the nature of the geometric progression of one dimensional. The nature of many properties of 1-D.G.P.(1) have not been developed for other types of geometric progressions discussed in the six new types. They can be propounded for them as the extension and research work. Here D.G.P. stands for dimensional geometric progression.

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