International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 14 Issue 05 May 2026, Page no. – 6393-6409
Index Copernicus ICV: 57.55, Impact Factor: 8.751
DOI: 10.47191/ijmcr/v14i5.06

Explicit Evaluations of Quintic theta function

[1]Ravi G. N., [2]Roja R. and [3]Praveenkumar

[1]Department of Mathematics, Government College for Women,
Kolar - 563101, INDIA.

[2]Department of studies in Mathematics, University of Mysore, Manasagangotri,
Mysuru - 570006, INDIA.

[3]Department of Mathematics, Government College for Women(Autonomous),
Mandya - 571401, INDIA.

Abstract

Recently, B. C. Berndt and Örs Rebák have obtained several evaluations of cubic theta function [7]. Motivated by their work, in this article we present several evaluations of quintic theta function identities.

Keywords : Ramanujan's theta functions, Values of theta functions, Class invariants, Quintic theta function, Ramanujan's lost notebook.

2020 Mathematics Subject Classification : 11F11, 11F27, 05A30, 33D15.

1 Introduction

The Ramanujan cubic theta function, also known as the Borwein's cubic theta function [11], is defined by

$$a\left(q\right)=\sum_{m,n=-\infty}^{\infty}q^{m^2+mn+n^2},\quad |q|<1.$$

Recently, B. C. Berndt and Örs Rebák [7] obtained several evaluations of

$$\frac{a(q^2)}{\varphi^2(q)}=\frac{\sum\limits_{m,n=-\infty}^{\infty}q^{2\left(m^2+mn+n^2\right)}}{\sum\limits_{m,n=-\infty}^{\infty}q^{m^2+n^2}},$$

where $\varphi(q)=\sum\limits_{n=-\infty}^{\infty}q^{n^2}$, one of the Ramanujan's theta function. For example

$$\frac{a(e^{-2\pi})}{\varphi^2(e^{-\pi})}=\left(\frac{1}{4}+\frac{1}{2\sqrt{3}}\right)^{1/4}.$$

For a detailed study on cubic theta function, one may refer [9, 6, 10, 12].

The quintic theta function $a_5(q)$ is defined by

$$a_5(q) := \sum_{n,m=-\infty}^{\infty} q^{2m^2+2mn+3n^2}.$$

By employing the famous Ramanujan's ${}_1\psi_1$ summation formula, A. Berkovick and H. Yesilyurt [2] have deduced

$$a_5(q) = 1 + \sum_{n=1}^{\infty} \left(\frac{-20}{n}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{1+q^{2n}},$$

where $\left(\frac{a}{n}\right)$ is a Legendre symbol for details see [14, p. 485-486]. Recently, K. R. Vasuki and P. Nagendra [18], have shown that

$$a_5(q) = \varphi(q^2)\varphi(q^{10}) + 4q^3\psi(q^4)\psi(q^{20}), \tag{1.1}$$

where $\varphi(q)$ and $\psi(q)$ are Ramanujan theta functions, defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Motivated by the works of Berndt and Rebák [7], in this paper we evaluate

$$\frac{a_5(q^{1/2})}{\varphi^2(q)}$$

for $q = e^{-\pi\sqrt{n}}$, where $n \in \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{5}, 1, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}, \frac{1}{45}\right\}$. Further, we also evaluate

$$\frac{a_5(q^{1/2})}{\varphi^2(q^5)}$$

for $q = e^{-\pi\sqrt{5n}}$, where $n \in \left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, 5, \frac{1}{5}, 9, \frac{1}{9}\right\}$.

2 Preliminary Results

As usual for any complex numbers a and q, let

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

Ramanujan defined his generalized theta function $f(a,b)$ in Chapter 16 of his second notebook [17] by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1.$$

From the Jacobi's triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

On the same page, Ramanujan also defined the following three special cases of $f(a,b)$ as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty.$$

Ramanujan also defines

$$\chi(-q) := (q;q^2)_\infty.$$

In the unorganized portion of the second notebook [6, p. 183], Ramanujan defined the following two class invariants G_n and g_n by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \tag{2.1}$$

and

$$g_n := 2^{-1/4}q^{-1/24}\chi(-q), \tag{2.2}$$

where $q = e^{(-\pi\sqrt{n})}$, n is a positive rational number. From [6, p. 187], we have

$$(g_nG_n)^8\left(G_n^8 - g_n^8\right) = \frac{1}{4} \tag{2.3}$$

and

$$g_{4n} = \sqrt[4]{2}g_nG_n. \tag{2.4}$$

We define, $f(-q^n){=}f_n$ for positive integer n. It is easy to see that

$$\varphi(q) = \frac{f_2^5}{f_1^2f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(-q) = \frac{f_1f_4}{f_2}, \quad f(q) = \frac{f_2^3}{f_1f_4},$$
$$\chi(-q) = \frac{f_1}{f_2} \quad \text{and} \quad \chi(q) = \frac{f_2^2}{f_1f_4}. \tag{2.5}$$

In our proof, we require the following known results:

Theorem 2.1. *For* $q = e^{-2\pi\sqrt{\frac{n}{5}}}$, *let*

$$J_n = \frac{\varphi(q)}{\sqrt[4]{5}\ \varphi(q^5)}. \tag{2.6}$$

Then

$$J_n J_{1/n} = 1, \tag{2.7}$$

$$J_1 = 1, \tag{2.8}$$

$$J_2 = \sqrt{6 - 4\sqrt{2} + 3\sqrt{5} - 2\sqrt{10}}, \tag{2.9}$$

$$J_{1/2} = \sqrt{2\sqrt{10} + 3\sqrt{5} - 4\sqrt{2} - 6}, \tag{2.10}$$

$$J_3 = \frac{1}{2}\sqrt{2\sqrt{5} - 2}, \tag{2.11}$$

$$J_{1/3} = \frac{1}{2}\sqrt{2\sqrt{5} + 2}, \tag{2.12}$$

$$J_4 = \frac{\sqrt{34 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}} - \sqrt{30 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}}}{2}, \tag{2.13}$$

$$J_{1/4} = \frac{\sqrt{34 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}} + \sqrt{30 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}}}{2}, \tag{2.14}$$

$$J_5 = \sqrt{5 - 2\sqrt{5}}, \tag{2.15}$$

$$J_{1/5} = \frac{\sqrt{5 + 2\sqrt{5}}}{\sqrt{5}}, \tag{2.16}$$

$$J_9 = \frac{(1 + \sqrt{3}\ )\ (\sqrt{5} - \sqrt{3}\ )}{2} \tag{2.17}$$

and

$$J_{1/9} = \frac{(\sqrt{3} - 1)\ (\sqrt{5} - \sqrt{3}\ )}{2}. \tag{2.18}$$

For a proof, see [13].

Theorem 2.2. *We have*

$$G_{3/5} = 2^{\frac{-1}{12}} \left(\sqrt{5} - 1\right)^{\frac{1}{3}} \tag{2.19}$$

and

$$G_{5/9} = \left(2 + \sqrt{5}\ \right)^{\frac{1}{4}} \left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}. \tag{2.20}$$

For a proof, see [1].

Theorem 2.3. *We have*

$$G_{10}G_{5/2} = \left(\frac{3+\sqrt{10}}{2}\right)^{\frac{1}{4}}, \tag{2.21}$$

$$G_{20}G_{5/4} = \frac{1}{2}\sqrt{\left(11+5\sqrt{5}\ \right)^{\frac{3}{4}} - 3\left(11+5\sqrt{5}\ \right)^{\frac{1}{4}} + a}, \tag{2.22}$$

$$\begin{aligned} G_{5/4} =& \frac{1}{2}\left(\left(11+5\sqrt{5}\right)^{\frac{1}{4}} - \sqrt{-4+\sqrt{11+5\sqrt{5}}}\ \right)^{\frac{1}{2}} \\ & \times\left(-3\left(11+5\sqrt{5}\right)^{\frac{1}{4}} + \left(11+5\sqrt{5}\right)^{\frac{3}{4}} + a\right)^{\frac{1}{4}}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} G_{20} =& \left(\left(11+5\sqrt{5}\right)^{\frac{1}{4}} - \sqrt{-4+\sqrt{11+5\sqrt{5}}}\ \right)^{\frac{-1}{2}} \\ & \times\left(-3\left(11+5\sqrt{5}\right)^{\frac{1}{4}} + \left(11+5\sqrt{5}\right)^{\frac{3}{4}} + a\right)^{\frac{1}{4}}, \end{aligned} \tag{2.24}$$

$$G_{5/2} = 2^{\frac{-1}{8}}\left(3+\sqrt{10}\right)^{\frac{1}{8}}\left(\left(\sqrt{5}+2\right)\left(5\sqrt{2}-7\right)\right)^{\frac{1}{12}} \tag{2.25}$$

and

$$G_{10} = 2^{\frac{-1}{8}}\left(3+\sqrt{10}\right)^{\frac{1}{8}}\left(\left(\sqrt{5}+2\right)\left(5\sqrt{2}-7\right)\right)^{\frac{1}{12}}, \tag{2.26}$$

where

$$a = \sqrt{16+\sqrt{11+5\sqrt{5}}\left(-3+\sqrt{11+5\sqrt{5}}\ \right)^{2}}.$$

For a proof, see [16].

Theorem 2.4. *We have*

$$g_n g_{4/n} = 1, \tag{2.27}$$

$$g_{20} = \left(1+\sqrt{5}\right)^{\frac{1}{24}}\left[2+\sqrt{5}+\sqrt{10+5\sqrt{5}}\ \right]^{\frac{1}{6}} = g_{1/5}^{-1} \tag{2.28}$$

and

$$g_5 = \frac{\left(2+\sqrt{5}+\sqrt{10+5\sqrt{5}}\ \right)^{\frac{1}{6}}}{\left(1+\sqrt{5}\ \right)^{\frac{1}{24}}} = g_{4/5}^{-1}. \tag{2.29}$$

For a proof, see [3].

Theorem 2.5. *We have*

$$G_n = G_{1/n}, \tag{2.30}$$

$$G_1 = 1, \tag{2.31}$$

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{4}}, \tag{2.32}$$

$$G_{15} = 2^{\frac{1}{4}}\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{3}}, \tag{2.33}$$

$$G_{25} = \frac{1+\sqrt{5}}{2}, \tag{2.34}$$

$$G_{45} = \left(2+\sqrt{5}\right)^{\frac{1}{4}}\left(\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right)^{\frac{1}{3}} \tag{2.35}$$

and

$$g_{10} = \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}}. \tag{2.36}$$

For a proof, see [6].

3 Evaluation of $\dfrac{a_5(q^{1/2})}{\varphi^2(q)}$

Theorem 3.1. *If n is a positive rational number then*

$$\frac{a_5\left(e^{\frac{-\pi}{2}\sqrt{\frac{n}{5}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{n}{5}}}\right)} = \frac{J_{1/n}}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{n/5}g_{5n})^2(G_{n/5}G_{5n})^4}\right).$$

Proof. From (1.1), we have

$$\frac{a_5(q^{1/2})}{\varphi^2(q)} = \frac{1}{\varphi^2(q)}\left(\varphi(q)\varphi(q^5)+4q^{\frac{3}{2}}\psi(q^2)\psi(q^{10})\right).$$

Using (2.5) to simplify the above, we obtain

$$\frac{a_5(q^{1/2})}{\varphi^2(q)} = \frac{\varphi(q^5)}{\varphi(q)}\left(1+\frac{4q^{\frac{3}{2}}\chi^2(-q)\chi^2(-q^5)}{\chi^4(-q^2)\chi^4(-q^{10})}\right).$$

Setting $q = e^{-\pi\sqrt{\frac{n}{5}}}$ in the above expression, and employing (2.1), (2.2), (2.4), (2.6) and (2.7), we obtain the required result. ☐

Corollary 3.1. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{\frac{-\pi}{\sqrt{5}}}\right)} = \frac{4+\left(\sqrt{5}-1\right)^{\frac{3}{2}}}{4\sqrt[4]{5}}.$$

Proof. Setting $n = 1$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{\frac{-\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{\frac{-\pi}{\sqrt{5}}}\right)} = \frac{J_1}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{1/5}g_5)^2(G_{1/5}G_5)^4}\right). \tag{3.1}$$

Using (2.30) and (2.32) to compute

$$G_{1/5}^4 G_5^4 = \frac{(3+\sqrt{5}\,)}{2}. \tag{3.2}$$

By setting $n = \frac{1}{5}$ and $n = 5$ in (2.3) and then solve for $g_{1/5}$ and g_5. We get

$$g_{1/5}^2 g_5^2 = \left(\frac{G_{1/5}^{12}-\sqrt{G_{1/5}^{24}-1}}{2G_{1/5}^4}\right)^{\frac{1}{4}}\left(\frac{G_5^{12}+\sqrt{G_5^{24}-1}}{2G_5^4}\right)^{\frac{1}{4}} = \frac{1}{\left(2\left(3+\sqrt{5}\,\right)\right)^{\frac{1}{4}}}. \tag{3.3}$$

Substituting the values from (3.2), (3.3), and (2.8) into (3.1), we obtain the required result. □

Corollary 3.2. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{\sqrt{10}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{2}{5}}}\right)} = \frac{(1+\sqrt{2}\,)\left(\sqrt{\sqrt{5}-2}\,\right)(\sqrt{10}-2)}{\sqrt[4]{5}}.$$

Proof. Setting $n = 2$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{\frac{-\pi}{\sqrt{10}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{2}{5}}}\right)} = \frac{J_{1/2}}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{2/5}g_{10})^2(G_{2/5}G_{10})^4}\right). \tag{3.4}$$

Using (2.21) and (2.30) . We get

$$G_{2/5}^4 G_{10}^4 = \frac{(3+\sqrt{10}\,)}{2}. \tag{3.5}$$

By setting $n = 10$ in $g_n g_{4/n} = 1$ and using (2.36). We obtain

$$g_{2/5} = \sqrt{\frac{\sqrt{5}-1}{2}}$$

and hence

$$g_{2/5}^2 g_{10}^2 = 1. \tag{3.6}$$

Substituting the values from (3.5), (3.6), and (2.10) into (3.4), we obtain the required result. ☐

Corollary 3.3. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{10}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{\sqrt{10}}}\right)} = \frac{(\sqrt{2}-1)\left(\sqrt{\sqrt{5}+2}\right)\left(1+\sqrt{2\sqrt{10}-6}\right)}{\sqrt[4]{5}}.$$

Proof. Setting $n = \frac{1}{2}$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{10}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{\sqrt{10}}}\right)} = \frac{J_2}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{1/10}g_{5/2})^2(G_{1/10}G_{5/2})^4}\right). \tag{3.7}$$

Using (2.21) and (2.30). We get

$$G_{2/5}^4 G_{10}^4 = \frac{\left(3+\sqrt{10}\right)}{2}. \tag{3.8}$$

By setting $n = \frac{5}{2}$ and $n = \frac{1}{10}$ in (2.3) and then solve for $g_{5/2}$ and $g_{1/10}$. Then on rationalising we get

$$\begin{aligned}\frac{1}{2(g_{1/10}g_{5/2})^2(G_{2/5}G_{10})^4} =& 2^{\frac{1}{4}}\left(\sqrt{10}-3\right)^{\frac{3}{4}}\left(G_{1/10}^{12}+\sqrt{G_{1/10}^{24}-1}\right)^{\frac{1}{4}} \\ &\times\left(G_{5/2}^{12}-\sqrt{G_{5/2}^{24}-1}\right)^{\frac{1}{4}},\end{aligned}$$

$$g_{1/10}^2 g_{5/2}^2 = \sqrt{2\sqrt{10}-6}. \tag{3.9}$$

Substituting the values from (3.8), (3.9), and (2.9) into (3.7), we obtain the required result. ☐

Corollary 3.4. *We have*

$$\frac{a_5\left(e^{-\pi\sqrt{\frac{3}{20}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{3}{5}}}\right)} = \left(4\sqrt{2}-\sqrt{3}+\sqrt{5}\right)\frac{\sqrt{\sqrt{5}+1}}{8\sqrt[4]{5}}.$$

Proof. Setting $n = 3$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\pi\sqrt{\frac{3}{20}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{3}{5}}}\right)} = \frac{J_{1/3}}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{3/5}g_{15})^2(G_{3/5}G_{15})^4}\right). \tag{3.10}$$

Using (2.19) and (2.33) to compute

$$G_{3/5}^4 G_{15}^4 = 4. \tag{3.11}$$

By setting $n = \dfrac{3}{5}$ and $n = 15$ in (2.3) and solve for $g_{3/5}^2$ and g_{15}^2 then using (2.27). We get

$$g_{3/5}^2 g_{15}^2 = \left(\frac{G_{3/5}^{12} - \sqrt{G_{3/5}^{24} - 1}}{2G_{3/5}^4} \right)^{\frac{1}{4}} \left(\frac{G_{15}^{12} + \sqrt{G_{15}^{24} - 1}}{2G_{15}^4} \right)^{\frac{1}{4}} = \frac{\sqrt{5} + \sqrt{3}}{2\sqrt{2}}. \tag{3.12}$$

Substituting the values from (3.11), (3.12), and (2.12) into (3.10), we obtain the required result. □

Corollary 3.5. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{15}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{\sqrt{15}}}\right)} = \left(4\sqrt{2} + \sqrt{3} + \sqrt{5}\right) \frac{\sqrt{\sqrt{5} - 1}}{8\sqrt[4]{5}}.$$

Proof. Setting $n = \dfrac{1}{3}$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{15}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{\sqrt{15}}}\right)} = \frac{J_3}{\sqrt[4]{5}} \left(1 + \frac{1}{2(g_{1/15} g_{5/3})^2 (G_{1/15} G_{5/3})^4}\right). \tag{3.13}$$

Using (2.19), (2.30) and (2.33) to compute

$$G_{5/3}^4 G_{1/15}^4 = 4. \tag{3.14}$$

By setting $n = \dfrac{1}{15}$ and $n = \dfrac{5}{3}$ in (2.3) and then solve for $g_{1/15}$ and $g_{5/3}$. We get

$$g_{1/15}^2 g_{5/3}^2 = \left(\frac{G_{1/15}^{12} - \sqrt{G_{1/15}^{24} - 1}}{2G_{1/15}^4} \right)^{\frac{1}{4}} \left(\frac{G_{5/3}^{12} + \sqrt{G_{5/3}^{24} - 1}}{2G_{5/3}^4} \right)^{\frac{1}{4}} = \frac{\sqrt{5} - \sqrt{3}}{2\sqrt{2}}. \tag{3.15}$$

Substituting the values from (3.14), (3.15), and (2.11) into (3.13), we obtain the required result. □

Corollary 3.6. *We have*

$$\begin{aligned} \frac{a_5\left(e^{-\frac{\pi}{\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{2\pi}{\sqrt{5}}}\right)} =& \frac{\left(\sqrt{34 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}} + \sqrt{30 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}}\right)}{4\sqrt[4]{5}} \\ &\times \left(2\left(1 + \sqrt{5}\right)^2 - \left(1 + \sqrt{5}\right)^{\frac{5}{2}}\right). \end{aligned}$$

Proof. Setting $n = 4$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{\pi}{\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{2\pi}{\sqrt{5}}}\right)} = \frac{J_{1/4}}{\sqrt[4]{5}}\left(1 + \frac{1}{2(g_{4/5}g_{20})^2(G_{4/5}G_{20})^4}\right). \tag{3.16}$$

Using (2.22) we get

$$G_{4/5}^4 G_{20}^4 = \frac{2\sqrt{5} + 6 + \sqrt{5}\sqrt{11 + 5\sqrt{5}}}{4}. \tag{3.17}$$

Using (2.28) and (2.29). We obtain

$$g_{4/5}^2 g_{20}^2 = \sqrt{1 + \sqrt{5}}. \tag{3.18}$$

Substituting the values from (3.17), (3.18), and (2.14) into (3.16), we obtain the required result. □

Corollary 3.7. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{4\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{2\sqrt{5}}}\right)} = \frac{\left(\sqrt{34 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}} - \sqrt{30 + 14\sqrt{5} - 8\sqrt{29 + 13\sqrt{5}}}\right)}{4\sqrt[4]{5}}$$
$$\times\left(2 + 2\left(1 + \sqrt{5}\right)^{\frac{7}{4}} - \left(1 + \sqrt{5}\right)^{\frac{9}{4}}\right).$$

Proof. Setting $n = \dfrac{1}{4}$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{\pi}{4\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{2\sqrt{5}}}\right)} = \frac{J_4}{\sqrt[4]{5}}\left(1 + \frac{1}{2(g_{1/20}g_{5/4})^2(G_{1/20}G_{5/4})^4}\right). \tag{3.19}$$

Using (2.22) and (2.30) we get

$$G_{5/4}^4 G_{1/20}^4 = \frac{2\sqrt{5} + 6 + \sqrt{5}\sqrt{11 + 5\sqrt{5}}}{4}. \tag{3.20}$$

By setting $n = \dfrac{1}{20}$ and $n = \dfrac{5}{4}$ in (2.3) and then solve for $g_{1/20}$ and $g_{5/4}$. We get

$$g_{1/20}^2 g_{5/4}^2 = \left(\frac{G_{1/20}^{12} - \sqrt{G_{1/20}^{24} - 1}}{2G_{1/20}^4}\right)^{\frac{1}{4}}\left(\frac{G_{5/4}^{12} + \sqrt{G_{5/4}^{24} - 1}}{2G_{5/4}^4}\right)^{\frac{1}{4}}. \tag{3.21}$$

Substituting the values from (2.23) and (2.24) into the above expression, and then substituting the values from (3.20), (3.21), and (2.13) into (3.19), we obtain the required result. □

Corollary 3.8. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{2}}\right)}{\varphi^2\left(e^{-\pi}\right)}=\left(\sqrt{2}+\left(\sqrt{5}-2\right)\left(161+12\left(6\sqrt{5}-\sqrt{360+161\sqrt{5}}\right)\right)^{\frac{1}{4}}\right)\times\left(\frac{\sqrt{2+\sqrt{5}}}{\sqrt{10}}\right).$$

Proof. Setting $n=5$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{\frac{-\pi}{2}}\right)}{\varphi^2\left(e^{-\pi}\right)}=\frac{J_{1/5}}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_1g_{25})^2(G_1G_{25})^4}\right). \tag{3.22}$$

Using (2.31) and (2.34) to compute

$$G_1^4G_{25}^4=\frac{7+3\sqrt{5}}{2^2}. \tag{3.23}$$

By setting $n=1$ and $n=25$ in (2.3) and then solve for g_1 and g_{25}. We get

$$g_1^2g_{25}^2=\left(\frac{G_1^{12}-\sqrt{G_1^{24}-1}}{2G_1^4}\right)^{\frac{1}{4}}\left(\frac{G_{25}^{12}+\sqrt{G_{25}^{24}-1}}{2G_{25}^4}\right)^{\frac{1}{4}},$$

$$g_1^2g_{25}^2=\frac{\left(\left(1+\sqrt{5}\right)^{12}+\sqrt{\left(1+\sqrt{5}\right)^{24}-2^{24}}\right)^{\frac{1}{4}}}{4\sqrt{2}\left(1+\sqrt{5}\right)}. \tag{3.24}$$

Substituting the values from (3.23), (3.24), and (2.16) into (3.22), we obtain the required result. □

Corollary 3.9. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{10}}\right)}{\varphi^2\left(e^{\frac{-\pi}{5}}\right)}=\left(\sqrt{2}+\left(\sqrt{5}-2\right)\left(161+12\left(6\sqrt{5}+\sqrt{360+161\sqrt{5}}\right)\right)^{\frac{1}{4}}\right)\times\left(\frac{\sqrt{\sqrt{5}-2}}{\sqrt{2}}\right).$$

Proof. Setting $n=\frac{1}{5}$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{\frac{-\pi}{10}}\right)}{\varphi^2\left(e^{\frac{-\pi}{5}}\right)}=\frac{J_5}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{1/25}g_1)^2(G_{1/25}G_1)^4}\right). \tag{3.25}$$

Using (2.30), (2.31) and (2.34) to compute

$$G_1^4 G_{1/25}^4 = \frac{7+3\sqrt{5}}{2^2}. \tag{3.26}$$

By setting $n = \dfrac{1}{25}$ and $n = 1$ in (2.3) and then solve for $g_{1/25}$ and g_1. We get

$$g_{1/25}^2 g_1^2 = \left(\frac{G_{1/25}^{12} - \sqrt{G_{1/25}^{24} - 1}}{2G_{1/25}^4}\right)^{\frac{1}{4}} \left(\frac{G_1^{12} - \sqrt{G_1^{24} - 1}}{2G_1^4}\right)^{\frac{1}{4}}$$

$$g_{1/25}^2 g_1^2 = \frac{\left(\left(1+\sqrt{5}\,\right)^{12} - \sqrt{\left(1+\sqrt{5}\,\right)^{24} - 2^{24}}\,\right)^{\frac{1}{4}}}{4\sqrt{2}\left(1+\sqrt{5}\,\right)}. \tag{3.27}$$

Substituting the values from (3.26), (3.27), and (2.15) into (3.25), we obtain the required result. □

Corollary 3.10. *We have*

$$\frac{a_5\left(e^{-\frac{3\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{3\pi}{\sqrt{5}}}\right)} = \left(\frac{2\sqrt{2} + \left(a - \sqrt{a^2-16}\,\right)^{\frac{1}{4}} \left(b - \sqrt{b^2-16}\,\right)^{\frac{1}{4}} \left(\sqrt{5}-2\right)^{\frac{3}{2}}}{\sqrt{2}\sqrt[4]{5}\left(1+\sqrt{3}\,\right)\left(\sqrt{5}-\sqrt{3}\,\right)}\right),$$

where $a = \left(2+\sqrt{5}\,\right)^3\left(\sqrt{5}-\sqrt{3}\,\right)^4$ *and* $b = \left(2+\sqrt{5}\,\right)^3\left(\sqrt{5}+\sqrt{3}\,\right)^4$.

Proof. Setting $n = 9$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{3\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{3\pi}{\sqrt{5}}}\right)} = \frac{J_{1/9}}{\sqrt[4]{5}}\left(1 + \frac{1}{2(g_{9/5}g_{45})^2(G_{9/5}G_{45})^4}\right). \tag{3.28}$$

Using (2.20), (2.30) and (2.35) we get

$$G_{9/5}^4 G_{45}^4 = 9 + 4\sqrt{5}. \tag{3.29}$$

By setting $n = \dfrac{9}{5}$ and $n = 45$ in (2.3) and then solve for $g_{9/5}$ and g_{45}. We get

$$g_{9/5}^2 g_{45}^2 = \left(\frac{G_{9/5}^{12} + \sqrt{G_{9/5}^{24} - 1}}{2G_{9/5}^4}\right)^{\frac{1}{4}} \left(\frac{G_{45}^{12} + \sqrt{G_{45}^{24} - 1}}{2G_{45}^4}\right)^{\frac{1}{4}}. \tag{3.30}$$

Substituting the values from (2.20) and (2.35) into the above expression, and then substituting the values from (3.29), (3.30), and (2.18) into (3.28), we obtain the required result. □

Corollary 3.11. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{6\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{3\sqrt{5}}}\right)}=\left(\frac{2\sqrt{2}+\left(a+\sqrt{a^2-16}\,\right)^{\frac{1}{4}}\left(b+\sqrt{b^2-16}\,\right)^{\frac{1}{4}}\left(\sqrt{5}-2\right)^{\frac{3}{2}}}{\sqrt{2}\sqrt[4]{5}\left(\sqrt{3}-1\right)\left(\sqrt{5}+\sqrt{3}\,\right)}\right),$$

where $a=\left(2+\sqrt{5}\,\right)^3\left(\sqrt{5}-\sqrt{3}\,\right)^4$ *and* $b=\left(2+\sqrt{5}\,\right)^3\left(\sqrt{5}+\sqrt{3}\,\right)^4.$

Proof. Setting $n=\dfrac{1}{9}$ in Theorem 3.1, we obtain

$$\frac{a_5\left(e^{-\frac{\pi}{6\sqrt{5}}}\right)}{\varphi^2\left(e^{-\frac{\pi}{3\sqrt{5}}}\right)}=\frac{J_9}{\sqrt[4]{5}}\left(1+\frac{1}{2(g_{1/45}g_{5/9})^2(G_{1/45}G_{5/9})^4}\right). \tag{3.31}$$

Using (2.20), (2.30) and (2.35) to compute

$$G_{5/9}^4G_{1/45}^4=9+4\sqrt{5}. \tag{3.32}$$

By setting $n=\dfrac{1}{45}$ and $n=\dfrac{5}{9}$ in (2.3) and then solve for $g_{1/45}$ and $g_{5/9}$. We get

$$g_{1/45}^2g_{5/9}^2=\left(\frac{G_{1/45}^{12}-\sqrt{G_{1/45}^{24}-1}}{2G_{1/45}^4}\right)^{\frac{1}{4}}\left(\frac{G_{5/9}^{12}-\sqrt{G_{5/9}^{24}-1}}{2G_{5/9}^4}\right)^{\frac{1}{4}}. \tag{3.33}$$

Substituting the values from (2.20) and (2.35) into the above expression, and then substituting the values from (3.32), (3.33), and (2.17) into (2.31), we obtain the required result. ☐

4 Evaluation of $\dfrac{a_5\left(q^{1/2}\right)}{\varphi^2(q^5)}$

Theorem 4.1. *If* n *is a positive rational number then*

$$\frac{a_5\left(e^{-\pi\sqrt{\frac{n}{20}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{5n}}\right)}=\sqrt[4]{5}J_n\left(1+\frac{1}{2(g_{n/5}g_{5n})^2(G_{n/5}G_{5n})^4}\right).$$

Proof. From (1.1), we have

$$\frac{a_5(q^{1/2})}{\varphi^2(q^5)}=\frac{1}{\varphi^2(q^5)}\left(\varphi(q)\varphi(q^5)+4q^{\frac{3}{2}}\psi(q^2)\psi(q^{10})\right).$$

Using (2.5) to simplify the above, we obtain

$$\frac{a_5(q^{1/2})}{\varphi^2(q^5)} = \frac{\varphi(q)}{\varphi(q^5)} \left(1 + \frac{4q^{\frac{3}{2}}\chi^2(-q)\chi^2(-q^5)}{\chi^4(-q^2)\chi^4(-q^{10})}\right).$$

Setting $q = e^{-\pi\sqrt{\frac{n}{5}}}$ in the above, and employing (2.1), (2.2), (2.4), (2.6) and (2.7), we obtain the required result. □

Corollary 4.1. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{5}}\right)} = \sqrt[4]{5}\left(1 + \frac{2}{\left(1+\sqrt{5}\right)^{\frac{3}{2}}}\right).$$

Corollary 4.2. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{\sqrt{10}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{10}}\right)} = \sqrt[4]{5}\left(\sqrt{2}-1\right)\left(\sqrt{\sqrt{5}+2}\right)\left(\sqrt{10}-2\right).$$

Corollary 4.3. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{10}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{5}{2}}}\right)} = \sqrt[4]{5}\left(\sqrt{2}+1\right)\left(\sqrt{\sqrt{5}-2}\right)\left(1+\sqrt{2\sqrt{10}-6}\right).$$

Corollary 4.4. *We have*

$$\frac{a_5\left(e^{-\pi\sqrt{\frac{3}{20}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{15}}\right)} = \sqrt[4]{5}\left(4\sqrt{2}-\sqrt{3}+\sqrt{5}\right)\frac{\sqrt{\sqrt{5}-1}}{8}.$$

Corollary 4.5. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{2\sqrt{15}}}\right)}{\varphi^2\left(e^{-\pi\sqrt{\frac{5}{3}}}\right)} = \sqrt[4]{5}\left(4\sqrt{2}+\sqrt{3}+\sqrt{5}\right)\frac{\sqrt{\sqrt{5}+1}}{8}.$$

Corollary 4.6. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{\sqrt{5}}}\right)}{\varphi^2\left(e^{-2\pi\sqrt{5}}\right)} = \frac{\sqrt[4]{5}\left(\sqrt{34+14\sqrt{5}-8\sqrt{29+13\sqrt{5}}} - \sqrt{30+14\sqrt{5}-8\sqrt{29+13\sqrt{5}}}\right)}{4} \times\left(2\left(1+\sqrt{5}\right)^2 - \left(1+\sqrt{5}\right)^{\frac{5}{2}}\right).$$

Corollary 4.7. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{4\sqrt{5}}}\right)}{\varphi^2\left(e^{\frac{-\pi\sqrt{5}}{2}}\right)}=\frac{\sqrt[4]{5}\left(\sqrt{34+14\sqrt{5}-8\sqrt{29+13\sqrt{5}}}+\sqrt{30+14\sqrt{5}-8\sqrt{29+13\sqrt{5}}}\right)}{4}\times\left(2+2\left(1+\sqrt{5}\right)^{\frac{7}{4}}-\left(1+\sqrt{5}\right)^{\frac{9}{4}}\right).$$

Corollary 4.8. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{2}}\right)}{\varphi^2\left(e^{-5\pi}\right)}=\left(\sqrt{2}+\left(\sqrt{5}-2\right)\left(161+12\left(6\sqrt{5}-\sqrt{360+161\sqrt{5}}\right)\right)^{\frac{1}{4}}\right)\times\left(\frac{\sqrt{5}\sqrt{\sqrt{5}-2}}{\sqrt{2}}\right).$$

Corollary 4.9. *We have*

$$\frac{a_5\left(e^{\frac{-\pi}{10}}\right)}{\varphi^2\left(e^{-\pi}\right)}=\left(\sqrt{2}+\left(\sqrt{5}-2\right)\left(161+12\left(6\sqrt{5}+\sqrt{360+161\sqrt{5}}\right)\right)^{\frac{1}{4}}\right)\times\left(\frac{\sqrt{\sqrt{5}+2}}{\sqrt{2}}\right).$$

Corollary 4.10. *We have*

$$\frac{a_5\left(e^{-\frac{3\pi}{2\sqrt{5}}}\right)}{\varphi^2\left(e^{-3\pi\sqrt{5}}\right)}=\left(\frac{\sqrt[4]{5}\left(1+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}{2}\right)\times\left(1+\frac{\left(a-\sqrt{a^2-16}\right)^{\frac{1}{4}}\left(b-\sqrt{b^2-16}\right)^{\frac{1}{4}}\left(\sqrt{5}-2\right)^{\frac{3}{2}}}{2\sqrt{2}}\right),$$

where $a=\left(2+\sqrt{5}\right)^3\left(\sqrt{5}-\sqrt{3}\right)^4$ *and* $b=\left(2+\sqrt{5}\right)^3\left(\sqrt{5}+\sqrt{3}\right)^4$.

Corollary 4.11. *We have*

$$\frac{a_5\left(e^{-\frac{\pi}{6\sqrt{5}}}\right)}{\varphi^2\left(e^{\frac{-\pi\sqrt{5}}{3}}\right)}=\left(\frac{\sqrt[4]{5}\left(\sqrt{3}-1\right)\left(\sqrt{5}+\sqrt{3}\right)}{2}\right)\times\left(1+\frac{\left(a+\sqrt{a^2-16}\right)^{\frac{1}{4}}\left(b+\sqrt{b^2-16}\right)^{\frac{1}{4}}\left(\sqrt{5}-2\right)^{\frac{3}{2}}}{2\sqrt{2}}\right),$$

where $a=\left(2+\sqrt{5}\right)^3\left(\sqrt{5}-\sqrt{3}\right)^4$ *and* $b=\left(2+\sqrt{5}\right)^3\left(\sqrt{5}+\sqrt{3}\right)^4$.

Proceeding with the same steps as in Corollaries (3.1)–(3.11) the proof of corollary (4.1)–(4.11) follows.

Acknowledgment

The authors would like to thank Prof. K. R. Vasuki for his advice and guidance during the preparation of this article.

References

[1] N. D. Baruah and P. Bhattacharyya, *Some theorems on the explicit evaluation of Ramanujan's theta-functions*, Int. J. Math. Math. Sci., **40**, 2149–2159 (2004).

[2] A. Berkovich and H. Yesilyurt, *Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms*, The Ramanujan J., **20**, 375–408 (2009).

[3] S. Bhargava, K. R. Vasuki and B. R. Srivatsa Kumar, *Evaluations of Ramanujan–Weber class invariant* g_n, J. Indian Math. Soc., **72**, (2005).

[4] N. D. Baruah and Nipen Saikia, *Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction*, J. of Comput and Appl Math., **160**, 37-51 (2003).

[5] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York (1991).

[6] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York (1998).

[7] B. C. Berndt and Örs Rebák, *Cubic and quintic analogues of Ramanujan's septic theta function identity*, Ramanujan J., **66**, 1-40 (2025).

[8] S. Bhargava, K. R. Vasuki and B. R. Srivatsa kumar, *Evaluations of Ramanujan-Weber class invariant* g_n, J. of the Indian Math. Soc., **72**, 115-127 (2005).

[9] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley and Sons, Inc (1987).

[10] H. H. Chan and S. Cooper, *Power of theta function*, Pac. J. Math., **235**(1), 1-14 (2008).

[11] H. H. Chan and L. Wang, *Borweins' cubic theta functions revisited*, Ramanujan J., **57**, 55–70 (2022).

[12] M. D. Hirschhorn, *A Simple proof of an identity of Ramanujan*, J. Austral. Math. Soc. (Series A)., **34**, 31-35 (1983).

[13] A. A. A. Kahtan, On Ramanujan's function $v(q) = R(q)R(q^4)$, *South East Asian J. Math. & Math. Sc.*, **9**, 1-26 (2011).

[14] T. Koshy, *Elementary number theory with applications*, Academic press, (2005).

[15] Z. G. Liu, *Some Eisenstein series identitiees related to modular equations of the seventh order*, Pac. J. Math., **209**(1), 103-130 (2003).

[16] Nipen Saikia, *A Parameter for Ramanujan's Function $\chi(q)$: Its Explicit Values and Applications*, ISRN Comput. Math., (2012).

[17] S. Ramanujan, Notebooks (2 volumes). *Tata institute of fundamental Research. Bombay*, (1957).

[18] K. R. Vasuki and P. Nagendra, *On certain relations among the generating functions for certain quadratic forms*, Notes Number Theory Discrete Math., **30**, 418–426 (2024).

[19] K. R. Vasuki and R. G. Veeresha, *Ramanujan's Eisenstein series of level 7 and 14*, J. Number Theory., **159**, 59-75 (2016).