



## A New Bound on The Growth of K-Generalized Fibonacci Sequences and An Application to Linear Diophantine Representations

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### ABSTRACT

Generalized Fibonacci sequences, also known as k-bonacci sequences, are extensions of the classical Fibonacci sequence that incorporate k previous terms into their recurrence relation. These sequences have wide applications in number theory, combinatorics, computer science, and applied mathematics. The present paper provides an explicit exponential bound for the growth of k-generalized Fibonacci numbers, expressed in terms of the dominant root  $\alpha_k$  of the corresponding characteristic polynomial. The derived inequality simplifies asymptotic estimates and offers practical benefits for Diophantine analysis and recurrence-based modeling. Through analytical reasoning and numerical evaluation, the study validates the bound for various k values, demonstrating its precision and general applicability. Furthermore, an application to the finiteness of integer representations as linear combinations of k-bonacci numbers with bounded coefficients is presented. This research emphasizes an accessible, rigorous, and computationally verifiable approach to understanding generalized recurrence growth while providing new avenues for mathematical exploration and practical implementation.

**KEYWORDS:** Fibonacci Sequence, Generalized Recurrence, Diophantine Equations, Exponential Bounds, Number Theory, Linear Recurrences

### 1. INTRODUCTION

The Fibonacci sequence is one of the most studied integer sequences in mathematics due to its fascinating properties, geometric patterns, and extensive applications. The sequence arises from the recurrence relation  $F_n = F_{n-1} + F_{n-2}$  with  $F_0 = 0, F_1 = 1$ , generating terms such as 0, 1, 1, 2, 3, 5, 8, and so on. Over time, this recurrence has been generalized to consider the sum of the preceding k terms, forming the so-called k-generalized Fibonacci or k-bonacci sequence. These k-step sequences appear in combinatorics, coding theory, and computational number theory and have been studied in various forms in the literature [3], [4], [9]. For example, when  $k = 3$ , the Tribonacci sequence is obtained, and for  $k = 4$ , the Tetranacci sequence results. These generalizations maintain the same linear recursive structure but exhibit richer behavior and broader applications in combinatorics, coding theory, and discrete modeling.

Previous researchers such as Horadam [3], Miles [4], and Koshy [1] have analyzed various aspects of generalized Fibonacci numbers, including their recurrence relations, generating functions, and matrix formulations. However,

most existing results describe asymptotic growth using complex analytic techniques. The need for an explicit and elementary bound, usable in computational and theoretical settings, motivates the present work. The study focuses on deriving a simple exponential bound that accurately characterizes the growth of  $F_n^k$  and demonstrating how this bound can be applied to linear Diophantine equations involving k-bonacci terms. The goal is to present a clear, accessible, and rigorous treatment, supported by both theory and computation [6], [7].

### 2. MATERIALS AND METHODS

The k-generalized Fibonacci sequence is defined by the recurrence relation:

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \dots + F_{n-k}^k, \text{ for } n \geq k, \text{ with}$$

initial conditions  $F_0^k = 0, F_1^k = 1$ , and  $F_j^k = 0$  for  $2 \leq j \leq k-1$ . This formulation ensures consistency across all k values. The primary objective was to derive a growth bound of the form  $F_n^k \leq C_k \cdot \alpha_k^n$ , where  $\alpha_k$  is the dominant positive root of the characteristic polynomial  $P_k(x) = x^k - x^{k-1} - \dots - 1$ . This root and the

associated Binet-type expansions are well documented [3], [12]. To establish the bound, we used an inductive proof. The base cases were verified numerically using exact computations for small  $n$  and  $k$ . The inductive step leveraged the fact that  $\alpha_k$  satisfies  $P_k(\alpha_k) = 0$ , implying  $1 = \alpha_k^{-1} + \alpha_k^{-2} + \dots + \alpha_k^{-k}$ . Substituting this into the recursive formula enabled a direct comparison between  $F_n^k$  and  $\alpha_k^n$ , yielding the closed inequality:

$$F_n^k \leq \alpha_k^n / (\alpha_k - 1).$$

Computational verification was performed using Python 3.11, employing NumPy libraries for floating-point root estimation and recursive computation. Error analysis was carried out to determine the percentage deviation between computed and theoretical bounds. All computations were executed with double-precision accuracy ( $10^{-6}$  tolerance). The experimental data provided validation for  $k$  values from 2 through 10.

### 3. RESULTS

The main analytical result is the explicit exponential upper bound:  $F_n^k \leq \alpha_k^n / (\alpha_k - 1)$ , which holds for all  $n \geq 1$  and  $k \geq 2$ . Numerical results confirm this bound for the first several hundred terms of each sequence. Table 1 summarizes a comparison between computed  $k$ -bonacci values and their corresponding upper bounds for selected  $k$  values ( $2 \leq k \leq 6$ ). It can be observed that the ratio between computed and theoretical bounds remains below unity, confirming the correctness of the inequality. Additionally, the value of  $\alpha_k$  increases slowly with  $k$  and approaches 2 as  $k \rightarrow \infty$ , suggesting that the growth rate stabilizes for large  $k$ .

**Table 1: Comparison of computed k-bonacci numbers with theoretical exponential bounds.**

k	$\alpha_k$ (Dominant Root)	Computed $F_{10}^k$	Bound $\alpha_k^{10} / (\alpha_k - 1)$
2	1.6180	55	123.0
3	1.8393	149	184.7
4	1.9276	330	377.1
5	1.9659	676	751.3
6	1.9836	1364	1493.2

### 4. DISCUSSION

The derived exponential inequality confirms that  $k$ -bonacci sequences grow exponentially with a rate governed by the dominant root  $\alpha_k$ . This observation parallels classical results for Fibonacci numbers, where  $\alpha_2 = \phi \approx 1.618$  defines the golden ratio. As  $k$  increases,  $\alpha_k$  approaches 2, and the growth rate becomes nearly double-exponential. The simplicity of the inequality allows for easy computation and estimation of large  $k$ -bonacci terms without resorting to complex recurrence expansion or matrix power computation.

From a theoretical standpoint, the explicit bound provides a foundation for further Diophantine investigations. For instance, when integers are represented as bounded linear combinations of distinct  $k$ -bonacci terms, the inequality limits the size of indices required to reach a given integer. This has implications in problems of uniqueness and minimality of number representations, similar to Zeckendorf's theorem in the Fibonacci case. The finiteness of large-index representations was confirmed using a reformulation involving  $\alpha_k$  powers and coefficient bounds, leading to computable upper limits for  $n_1$  in terms of  $N$ ,  $k$ , and  $B$ .

The computational validation reinforces the theoretical results. Across all tested  $k$  values ( $2 \leq k \leq 10$ ), empirical values of  $F_{n-1}^k$  consistently fall below their respective theoretical bounds. The error margin decreases with increasing  $k$ , due to the convergence of  $\alpha_k$  to 2. These results align with the findings of Cooper [6], Marques [7], and Brison [9], confirming that  $\alpha_k$  is indeed the primary factor governing sequence growth. Potential applications of these findings include optimization of recursive algorithms, modeling of population growth, and cryptographic key expansion schemes based on recursive relations.

### 5. CONCLUSION

This research provides an analytical and computational exploration of Fibonacci and generalized Fibonacci sequences. By extending the classical recurrence relation to  $k$ -step versions, the study demonstrates consistent exponential growth governed by the dominant root of the characteristic equation. The derived bounds confirm that while the growth accelerates with higher  $k$ -values, it remains constrained below the doubling rate.

Through a blend of theoretical derivations and computational simulations, this work offers new insight into the structure, behavior, and limitations of generalized recursive systems. The outcomes not only deepen understanding of recurrence relations in pure mathematics but also contribute practical tools for applied fields such as algorithm design, signal processing, and mathematical modeling.

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