International Journal of Mathematics and Computer Research

ISSN: 2320-7167

Volume 14 Issue 02 February 2026, Page no. – 6152-6160

Index Copernicus ICV: 57.55, Impact Factor: 8.615

DOI: 10.47191/ijmcr/v14i2.02

Generalized $\alpha - E$ Suzuki Type Contractions and Fixed-Point Theorems in Metric-Like Spaces

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ARTICLE INFO	ABSTRACT
Published Online: 02 February 2026 **Corresponding Author:** Kuldeep Kumar Tiwari	In this manuscript, we shall give new notion of generalized $\alpha - E$ −type contractions and prove related fixed point theorems in complete metric-like space. Then, in the form of corollaries some consequences of our proved results will be provided. Finally, an example will be given to show the real existence of our proved results.

KEYWORDS: Keywords: fixed point, generalized $\alpha - E$ −contractions, metric-like space.

MSC 2020: 47H10, 54H25, 65Q10.

1. INTRODUCTION

Metrical fixed point theory is one of major branch of research in non-linear analysis. Banach [5] contraction principle is the main pillar of metrical fixed point theory which states that “Every contraction map on a complete metric space has a unique fixed point.” Many authors proved fixed point theorems in various spaces see ([1-3], [10-15]).

To get new fixed point results in 2012, Amini Harandi [3] introduced metric-like space.

Definition 1.1. [3] A mapping $d : X \times X \to \mathbb{R}^+$ where X is a nonempty set, is said to be metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

(i) $d(x,y) = 0 \Rightarrow x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

The pair (X,d) is then called a metric-like (dislocated) space.

Definition 1.2. [3] Consider a metric like space (X,d).

1. If for the sequence (x_n) in X $\lim_{m,n\to\infty} d(x_n,x_m)$ exists and finite then it is said to be Cauchy.
2. If every Cauchy sequence (x_n) in X converges to some $x \in X$ then (X,d) is said to be complete, that is $\lim_{n\to\infty} d(x,x_n) = d(x,x) = \lim_{m,n\to\infty} d(x_n,x_m)$.
3. Let us consider a self map $T : (X,d) \to (X,d)$, if for any sequence (x_n) in X such that $d(x,x_n) \to d(x,x)$ as $n \to \infty$, we have $d(Tx,Tx_n) \to d(x,x)$ as $n \to \infty$, then the map is continuous.

Wardowski [4] established the concept of E-contraction as follows.

Definition 1.3. [4] A collection of functions $E \in F$ mapping from $[0,\infty)$ into $(-\infty,+\infty)$ has the following conditions:

$(\mathcal{E}_1)$ E is strictly increasing; that is, $\forall \eta, \mu \in [0,\infty)$ such that $\eta < \mu, E(\eta) < E(\mu)$.

$(\mathcal{E}_2)$ For each sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of $[0,\infty)$, $\lim_{n\to\infty} \eta_n = 0 \Leftrightarrow \lim_{n\to\infty} E(\eta_n) = -\infty$.

$(\mathcal{E}_3)$ For every $\kappa \in (0,1)$, $\lim_{n\to\infty} \eta^k E(\eta_n) = 0$.

Definition 1.4. [4] A self-map T on X is called an E −contraction if $\exists\, E \in F$ and $b > 0$ such that $\forall x, y \in X$,

$d(Tx,Ty) > 0$ gives $b + E\big(d(Tx,Ty)\big) \leq E\big(d(x,y)\big)$.

Edelstein [7] demonstrated a version of the following theorem in 1962.

Theorem 1.5. [7] Consider a self-map T on X. Assume that $d(Tx,Ty) < d(x,y)$ is true $\forall\, x, y \in X$ with $x \neq y$.

Then T has a unique fixed point.

Suzuki, in 2008 [14], showed that Edelstein's conclusions could be generalized in a compact metric space.
Theorem 1.6. Let T be a self-map on a compact metric space X. Assume that $d(Tx,Ty) < d(x,y)$ is true $\forall\, x,y \in X$ with $x \neq y$, $\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) < d(x,y)$.
Then T has a unique fixed point in X.
Wardowski [4] demonstrated a version of the following theorem in 2012.
Theorem 1.7. Consider an E-contractive self-map T on a complete metric space X. Then, we have the following:
1. T has a unique fixed point y in X;
2. For all $x \in X$, the sequence $\{T^n(x)\}$ is convergent to y in X.

Secelean [12] demonstrated the following lemma.
Lemma 1.8. Consider an increasing mapping $E : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and a sequence $\{\eta_n\}_{n=1}^{\infty}$ of $\mathbb{R}^+$. Then the following conditions hold:
1. If $\lim_{n\to\infty} E(\eta_n) = -\infty$, then $\lim_{n\to\infty} \eta_n = 0$;
2. If $infimum(E) = -\infty$, and $\lim_{n\to\infty} \eta_n = 0$, then $\lim_{n\to\infty} E(\eta_n) = -\infty$.

Secelean established that the condition $(\mathcal{E}_2)$ in Definition 1.3 may be retrieved by establishing Lemma 1.8, and Hossan Piri and Poom Kumam [10] demonstrated that the condition $(\mathcal{E}_3)$ in Definition 1.3 may be retrieved by a condition that is comparable but simpler.
Definition 1.9. [10] The set of all functions $E : \mathbb{R}^+ \longrightarrow \mathbb{R}$ fulfils the following:
$(\mathcal{E}_1)$ E is strictly increasing; that is, $\forall\, \eta, \mu \in \mathbb{R}^+$ such that $\eta < \mu$, $E(\eta) < E(\mu)$.
$(\mathcal{E}'_2)$ Infimum $(E) = -\infty$. (or)
$(\mathcal{E}''_2)$ There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of $\mathbb{R}^+$ such that $\lim_{l\to\infty} E(\eta_l) = -\infty$.
$(\mathcal{E}'_3)$ E is continuous on $\mathbb{R}^+$.
Example 1.10. [10] Let $E_1(\eta) = -\frac{1}{\eta}$, $E_2(\eta) = -\frac{1}{\eta} + \eta$, $E_1(\eta) = \frac{1}{1-e^{\eta}}$. Then $E_1, E_2, E_3 \in F$.
Definition 1.11. [10] Let T be a self-map on a metric space (X,d), which is known as orbitally continuous on X if $\lim_{n\to\infty} T^{l_n}(x) = y \Rightarrow \lim_{l\to\infty} T^{l_n}(x) = Ty$.
Let T be a self-map on X.
We denote $\mathrm{Fix}(T) = \{x : Tx = x\, \forall\, x \in X\}$.
Definition 1.12. [11] Consider a self-map T on X and a mapping $\alpha : X \times X \to [0,\infty)$; then T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$.
Definition 1.13. [9] Let T be a self-map on X and consider a mapping $\alpha : X \times X \to (-\infty,+\infty)$. Then T is triangular α-admissible if
1. $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$;
2. $\alpha(x,ÎÎ) \geq 1$, and $\alpha(ÎÎ,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$

for all $x,y,ÎÎ \in X$.
Example 1.14. Let $T : [0,\infty) \to [0,\infty)$ and $\alpha : [0,\infty) \times [0,\infty) \to (-\infty,+\infty)$ defined by $Tx = In(1+x)\, \forall\, x \in [0,\infty)$ and $\alpha(x,y) = \begin{cases} 1+x, if\ x \geq y; \\ 0, else \end{cases}$.
Then, by Definition 1.12
$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1\, \forall\, x \geq y$ and $\alpha(x,y) = \alpha(y,x)\, \forall\, x = y$.
Definition 1.15. [8] Let X be a non-void set and let T be an α-admissible map on X.
Then, T has the following condition:
(H) if each $x,y \in Fix(Q), \exists\, ÎÎ \in T$ such that $\alpha(x,ÎÎ) \geq 1$ and $\alpha(y,ÎÎ) \geq 1$.
Definition 1.16. [13] Let X be a non-void set and let T be an $\alpha-$admissible map on X. Then T is $\alpha^* -$ admissible if $\forall\, x,y \in Fix(Q) \neq \emptyset$, we have $\alpha(x,y) \geq 1$.
If $Fix(Q) = \emptyset$, we say that T is vacuously $\alpha^* -$ admissible.
Some authors, such as Alsulami *et al.* [20], Khan *et al.* [24], and others, have used the idea above without its nomenclature concerning the uniqueness of the fixed point.
Example 1.17. Define $T : [0,\infty) \to [0,\infty)$ and $\alpha : [0,\infty) \times [0,\infty) \to (-\infty,+\infty)$ by $Tx = 1 + x\, \forall\, x \in [0,\infty)$ and $\alpha(x,y) = \begin{cases} e^{2(x-y)}\ if\ x \geq y \\ 0, else \end{cases}$.
Then, T is $\alpha-$admissible.
Here, T is vacuously $\alpha^* -$ admissible because Q has no fixed point; that is, $Fix(T) = \emptyset$.
Definition 1.18. [15] We say that a self-map T on X is an $\alpha - E -$contraction if a function $\alpha : X \times X \to [0,\infty)$ exists and $E \in F$ such that

$d(Tx,Ty) > 0 \Rightarrow b + E\big(\alpha(x,y)d(Tx,Ty)\big) \le E\big(d(x,y)\big), \forall\, x,y \in X.$

Theorem 1.19. [15] Let (X,d) be a complete metric space and $T : X \to X$ be an $\alpha - E$ −contraction, the subsequent assertion hold:

(i) T is $\alpha -$ admissible;

(ii) $\exists\, x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$.

(iii) T is continuous or orbitally continuous on X.

Then, the point $\text{Ⅱ} \in X$ is a fixed point of T.

Moreover, if T is α^* −admissible, then the point $\text{Ⅱ} \in X$ is a unique fixed point of T. Furthermore, for every $x_0 \in X$ if $x_{l+1} = T^{l+1}x_0 \ne Tx_l, \forall l \ge 0$, then $\lim_{n\to\infty} T^l x_0 = \text{Ⅱ}$.

Definition 1.20. [15] A self-map T on X is called an $\alpha - E$ −Suzuki contraction if $b > 0$ exists such that $\forall\, x,y \in X$ with $Tx \ne Ty$

$\frac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow b + E\big(\alpha(x,y)d(Qx,Qy)\big) \le E\big(d(x,y)\big)$

Where $E \in F$.

Theorem 1.21. [15] Let (X,d) be a complete metric space and $T : X \to X$ be an $\alpha - E$ −Suzuki contraction, the subsequent assertion hold:

(i) T is $\alpha -$ admissible;

(ii) $\exists\, x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$.

(iii) T is continuous or orbitally continuous on $T.\text{ⱧA}$

Then, the point $\text{Ⅱ} \in X$ is a fixed point of ⱧA. Moreover, if ⱧA is α^* −admissible, then the point $\text{Ⅱ} \in X$ is a unique fixed point of ⱧA. Furthermore, for every $x_0 \in X$ if $x_{l+1} = \text{ⱧA}^{l+1}x_0 \ne \text{ⱧA}x_l, \forall l \ge 0$, then $\lim_{n\to\infty} \text{ⱧA}^l x_0 = \text{Ⅱ}$.

Now, we are ready to define generalized $\alpha - E$ −contractions and prove related fixed point theorems.

2. MAIN RESULTS

In this section of the paper, we define the class of generalized $\alpha - E$ −contractions.

Definition 2.1. We define that a self-map ⱧA on metric-like space X is a generalized $\alpha - E$-contraction if a function $\alpha : X \times X \to [0,\infty)$ exists and $E \in F, b > 0$ such that

$$\mathfrak{h}(\text{ⱧA}\tilde{\omega}\varsigma, \text{ⱧA}y) > 0 \Rightarrow b + E\big(\alpha(\tilde{\omega},y)\mathfrak{h}(\text{ⱧA}\tilde{\omega}, \text{ⱧA}y)\big) \le E\big(M(\tilde{\omega},y)\big), \quad (2.1)$$

Where $M(\tilde{\omega},y) = \{\mathfrak{h}(\tilde{\omega},y), \mathfrak{h}(\tilde{\omega},\text{ⱧA}\tilde{\omega}), \mathfrak{h}(y,\text{ⱧA}y)\}, \forall\, \tilde{\omega}, y \in \omega$.

Lemma 2.2. Suppose ⱧA is a generalized $\alpha - E$ −contractive mapping on metric-like space $(\omega, \mathfrak{h})$, the below situations holds:

(i) ⱧA is $\alpha -$ admissible;

(ii) $\exists\, \tilde{\omega}_0 \in \omega$ s.t. $\alpha(\tilde{\omega}_0, \text{ⱧA}\tilde{\omega}_0) \ge 1$.

Also, construct a sequence $\{\tilde{\omega}_l\} \in \omega$ by $\tilde{\omega}_{l+1} = \text{ⱧA}^{l+1}\tilde{\omega}_0 = \text{ⱧA}\tilde{\omega}_l, \forall l \in \mathbb{N} \cup \{0\}$.

Then, $\alpha(\tilde{\omega}_l, \tilde{\omega}_{l+1}) > 1, \forall l \ge 0$ and $E\big(\mathfrak{h}(\tilde{\omega}_l, \text{ⱧA}\tilde{\omega}_l)\big) = E\big(\alpha(\tilde{\omega}_{l-1}, \tilde{\omega}_l)\mathfrak{h}(\tilde{\omega}_l, \text{ⱧA}\tilde{\omega}_l)\big) \le E\big(\mathfrak{h}(\tilde{\omega}_0, \text{ⱧA}\tilde{\omega}_0)\big) - lb$.

Proof. Let $\tilde{\omega}_0 \in \omega$. It follows that $\alpha(\text{ⱧA}\tilde{\omega}_0, \tilde{\omega}_0) \ge 1$, and we construct a sequence $\{\tilde{\omega}_l\}$ by $\tilde{\omega}_{l+1} = \text{ⱧA}^{l+1}\tilde{\omega}_0 = \text{ⱧA}\tilde{\omega}_l, \forall l \in \mathbb{N} \cup \{0\}$.

By Definition 1.12, we get

$\alpha(\tilde{\omega}_0, \tilde{\omega}_1) = \alpha(\tilde{\omega}_0, \text{ⱧA}\tilde{\omega}_0) \ge 1 \Longrightarrow \alpha(\tilde{\omega}_1, \tilde{\omega}_2) = \alpha(\text{ⱧA}\tilde{\omega}_0, \text{ⱧA}^2\tilde{\omega}_0) \ge 1.$

Consequently, we got inductively that $\alpha(\tilde{\omega}_l, \tilde{\omega}_{l+1}) > 1, \quad \forall\ l \ge 0$.

Let $\tilde{\omega}_l \ne \tilde{\omega}_{l+1} \quad \forall\ l \ge 0$.

So $\mathfrak{h}(\tilde{\omega}_l, \tilde{\omega}_{l+1}) > 0, \quad \forall \quad l \ge 0$.

Using the condition $\mathcal{E}_1$ and $\alpha(\tilde{\omega}_0, \tilde{\omega}_1) \ge 1$ by equation (2.1), we get

$$\begin{aligned} E\big(\mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3)\big) &= E\big(\mathfrak{h}(\text{ⱧA}\tilde{\omega}_1, \text{ⱧA}\tilde{\omega}_2)\big) \\ &\le E\big(\alpha(\tilde{\omega}_1, \tilde{\omega}_2)\mathfrak{h}(\text{ⱧA}\tilde{\omega}_1, \text{ⱧA}\tilde{\omega}_2)\big) \\ &\le E\big(M(\tilde{\omega}_1, \tilde{\omega}_2)\big) - b \qquad (2.2) \end{aligned}$$

where

$$\begin{aligned} M(\tilde{\omega}_1, \tilde{\omega}_2) &= max\{\mathfrak{h}(\tilde{\omega}_1, \tilde{\omega}_2), \mathfrak{h}(\tilde{\omega}_1, \text{ⱧA}\tilde{\omega}_1), \mathfrak{h}(\tilde{\omega}_2, \text{ⱧA}\tilde{\omega}_2)\}, \\ &= max\{\mathfrak{h}(\tilde{\omega}_1, \tilde{\omega}_2), \mathfrak{h}(\tilde{\omega}_1, \tilde{\omega}_2), \mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3)\}. \end{aligned}$$

Now, if possible, suppose that $\mathfrak{h}(\tilde{\omega}_1, \tilde{\omega}_2) < \mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3)$.

Then, equation (2.2) becomes

$E\big(\mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3)\big) < E\big(\mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3)\big).$

A contradiction.

Thus,

$\mathfrak{h}(\tilde{\omega}_2, \tilde{\omega}_3) < \mathfrak{h}(\tilde{\omega}_1, \tilde{\omega}_2).$

Again, by equation (2.1), we have

$E\big(\mathfrak{h}(\tilde{\omega}_3, \tilde{\omega}_4)\big) = E\big(\mathfrak{h}(\text{ⱧA}\tilde{\omega}_2, \text{ⱧA}\tilde{\omega}_3)\big)$

$$\leq E\big(\alpha(\hat{\omega}_2,\hat{\omega}_3)\hbar(\mathbb{H}\hat{\omega}_2,\mathbb{H}\hat{\omega}_3)\big)$$
$$\leq E\big(M(\hat{\omega}_2,\hat{\omega}_3)\big)-b \qquad (2.3)$$

where

$$M(\hat{\omega}_2,\hat{\omega}_3)=max\{\hbar(\hat{\omega}_2,\hat{\omega}_3),\hbar(\hat{\omega}_2,\mathbb{H}\hat{\omega}_2),\hbar(\hat{\omega}_3,\mathbb{H}\hat{\omega}_3)\},$$
$$=max\{\hbar(\hat{\omega}_2,\hat{\omega}_3),\hbar(\hat{\omega}_2,\hat{\omega}_3),\hbar(\hat{\omega}_3,\hat{\omega}_4)\}.$$

Now, if possible, suppose that $\hbar(\hat{\omega}_1,\hat{\omega}_2)<\hbar(\hat{\omega}_2,\hat{\omega}_3)$.

Then, equation (2.3) becomes

$$E\big(\hbar(\hat{\omega}_3,\hat{\omega}_4)\big)<E\big(\hbar(\hat{\omega}_3,\hat{\omega}_4)\big).$$

A contradiction.

Thus,

$$\hbar(\hat{\omega}_3,\hat{\omega}_4)<\hbar(\hat{\omega}_2,\hat{\omega}_3).$$

Through the use of this approach, we are able to demonstrate inductively that $\mathbb{H}$ has a sequence that is strictly non-increasing $\{\hbar(\hat{\omega}_l,\hat{\omega}_{l+1})\}$ in ω.

Theorem 2.3. Let $(\omega,\hbar)$ be a complete metric-like space and $\mathbb{H}:\omega\to\omega$ be a generalized $\alpha-E$ −contraction, the subsequent assertion hold:

1. $\mathbb{H}$ is $\alpha-$ admissible;
2. $\exists\, \hat{\omega}_0\in\omega$ such that $\alpha(\hat{\omega}_0,\mathbb{H}\hat{\omega}_0)\geq 1$;
3. $\alpha(\hat{\omega},\hat{\omega})\geq 1$, for all $\hat{\omega}\in\omega$;
4. $\mathbb{H}$ is continuous or orbitally continuous on ω.

Then, the point $\mathrm{ll}\in\omega$ is a fixed point of $\mathbb{H}$.

Moreover, if $\mathbb{H}$ is α^* −admissible, then the point $\mathrm{ll}\in\omega$ is a unique fixed point of $\mathbb{H}$. Furthermore, for every $\hat{\omega}_0\in\omega$ if $\hat{\omega}_{l+1}=\mathbb{H}^{l+1}\hat{\omega}_0\neq\mathbb{H}\hat{\omega}_l,\forall l\geq 0$, then $\lim_{n\to\infty}\mathbb{H}^l\hat{\omega}_0=\mathrm{ll}$.

Proof: Let $\hat{\omega}_0\in\omega$ be such that $\alpha(\mathbb{H}\hat{\omega}_0,\hat{\omega}_0)\geq 1$ and construct a sequence $\{\hat{\omega}_l\}$ by $\hat{\omega}_{l+1}=\mathbb{H}^{l+1}\hat{\omega}_0=\mathbb{H}\hat{\omega}_l,\forall\, l\geq 0$. If $\hat{\omega}_{l_0}=\hat{\omega}_{l_0+1}$, it follow that $\mathbb{H}\hat{\omega}_{l_0}=\hat{\omega}_{l_0}$ for few $l_0\geq 0$; so $\hat{\omega}_{l_0}$ in ω is a fixed point of $\mathbb{H}$.

Then, Let $\hat{\omega}_l\neq\hat{\omega}_{l+1}\forall\, l\geq 0$. Then, $\hbar(\hat{\omega}_l,\hat{\omega}_{l+1})>0\ \forall\, l\geq 0$.

Using Lemma 2.2, we get

$$\alpha(\hat{\omega}_l,\hat{\omega}_{l+1})>1\quad\forall\quad l\geq 0.$$

Then, $\alpha(\hat{\omega}_l,\hat{\omega}_{l+1})=\alpha(\mathbb{H}^l\hat{\omega}_0,\mathbb{H}^{l+1}\hat{\omega}_0)\geq 1\ \forall l\geq 0$.

According to $\mathfrak{d}_1$ and Definition 1.12

$$\frac{1}{2}\hbar(\hat{\omega}_l,\mathbb{H}\hat{\omega}_l)=\frac{1}{2}\hbar(\hat{\omega}_l,\hat{\omega}_{l+1}),$$
$$<\hbar(\hat{\omega}_l,\hat{\omega}_{l+1}).$$

Now, by equation (1), we get

$$E\big(\hbar(\hat{\omega}_l,\hat{\omega}_{l+1})\big)=E\big(\hbar(\mathbb{H}\hat{\omega}_{l-1},\mathbb{H}\hat{\omega}_l)\big)$$
$$=E\big(\alpha(\hat{\omega}_{l-1},\hat{\omega}_l)\hbar(\mathbb{H}\hat{\omega}_{l-1},\mathbb{H}\hat{\omega}_l)\big)$$
$$\leq E\big(\hbar(\hat{\omega}_{l-1},\hat{\omega}_l)\big)-b \qquad (2.4)$$

Repeating this process, we get

$$E\big(\hbar(\mathbb{H}\hat{\omega}_{l-1},\mathbb{H}\hat{\omega}_l)\big)\leq E\big(\hbar(\hat{\omega}_{l-1},\hat{\omega}_l)\big)-b$$
$$=E\big(\hbar(\mathbb{H}\hat{\omega}_{l-2},\mathbb{H}\hat{\omega}_{l-1})\big)-b$$
$$=E\big(\alpha(\hat{\omega}_{l-2},\hat{\omega}_{l-1})\hbar(\mathbb{H}\hat{\omega}_{l-2},\mathbb{H}\hat{\omega}_{l-1})\big)-b$$
$$\leq E\big(\hbar(\hat{\omega}_{l-2},\hat{\omega}_{l-1})\big)-2b$$
$$=E\big(\hbar(\mathbb{H}\hat{\omega}_{l-3},\mathbb{H}\hat{\omega}_{l-2})\big)-2b$$
$$=E\big(\alpha(\hat{\omega}_{l-3},\hat{\omega}_{l-2})\hbar(\mathbb{H}\hat{\omega}_{l-3},\mathbb{H}\hat{\omega}_{l-2})\big)-2b$$
$$\leq \mathfrak{d}\big(\hbar(\hat{\omega}_{l-3},\hat{\omega}_{l-2})\big)-3b$$
$$.$$
$$.$$
$$.$$
$$\leq E\big(\hbar(\hat{\omega}_0,\hat{\omega}_1)\big)-lb. \qquad (2.5)$$

By applying limit on every sides, we get

$$\lim_{n\to\infty}E\big(\hbar(\mathbb{H}\hat{\omega}_{l-1},\mathbb{H}\hat{\omega}_l)\big)=-\infty. \qquad (2.6)$$

Therefore, by $(\mathfrak{d}_2)$ of Definition 2.3 with equation (2.6), we get

$$\lim_{l\to\infty}\hbar(\mathbb{H}\hat{\omega}_{l-1},\mathbb{H}\hat{\omega}_l)=0. \qquad (2.7)$$

By $(\mathfrak{d}_3)$ of Definition 2.3, $\exists k$ in $(0,1)$ such that

$$\lim_{l\to\infty}\left(\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})\right)^k E\left(\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})\right)=0. \qquad (2.8)$$

Also, by equation (2.5), we have

$$[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k\left[E\left(\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})\right)-E\left(\mathfrak{h}(\hat{\omega}_0,\hat{\omega}_1)\right)\right]\le -[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k lb\le 0. \qquad (2.9)$$

Using $l\to\infty$ in the previous equation together with the two different equations (2.7) and (2.8), we have

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k=0. \qquad (2.10)$$

Then, we will solve two different cases.

Case (i): Consider l is a multiple of 2; by Equation (2.10), we get

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k=0. \qquad (2.11)$$

Case(ii): Consider l is not an multiple of 2; by Equation (2.10), we get

$$\lim_{l\to\infty}(l-1)[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k=0. \qquad (2.12)$$

By Equations (2.7) and (2.12), we have

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k=0. \qquad (2.13)$$

From the previous equations we see that $\exists\ l_1\in\mathbb{N}$ such that

$l[\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})]^k\le 1\ \forall\, l\ge l_1$.

Therefore, we have

$\mathfrak{h}(\hat{\omega}_l,\hat{\omega}_{l+1})\le \frac{1}{l^{\frac{1}{k}}}\ \ \forall\, l\ge l_1$.

Further, we will shows that sequence $\{\hat{\omega}_l\}$ is an Cauchy.

Now, $\forall\, p>r\ge l_1$, we have

$$\mathfrak{h}\left(\hat{\omega}_p,\hat{\omega}_r\right)\le \mathfrak{h}\left(\hat{\omega}_p,\hat{\omega}_{p-1}\right)+\mathfrak{h}\left(\hat{\omega}_{p-1},\hat{\omega}_{p-2}\right)+\mathfrak{h}\left(\hat{\omega}_{p-2},\hat{\omega}_{p-3}\right)+\cdots+\mathfrak{h}\left(\hat{\omega}_{r+1},\hat{\omega}_r\right)$$

$$<\sum_{n=r}^{\infty}\mathfrak{h}(\hat{\omega}_n,\hat{\omega}_{n+1})$$

$$\le \textstyle\sum_{n=r}^{\infty}\frac{1}{n^{\frac{1}{k}}}.$$

Taking limit $r\to\infty$, we get $\lim_{p,r\to\infty}\mathfrak{h}\left(\hat{\omega}_p,\hat{\omega}_r\right)=0$, Since $\le \sum_{n=r}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent if $k<1$. This proves that sequence $\{\hat{\omega}_l\}$ in Ꞷ is an Cauchy.

So Ꞷ is complete, we will have $\text{ƚƚ}\in\text{Ꞷ}$ s.t.

$$\lim_{l\to\infty}\mathfrak{h}(\text{ƚƚ},\hat{\omega}_l)=\mathfrak{h}(\text{ƚƚ},\text{ƚƚ})=\lim_{m,l\to\infty}\mathfrak{h}(\hat{\omega}_m,=\hat{\omega}_l)=0. \qquad (2.14)$$

Since ȽA is continuous, from equation (2.14), we have

$$\lim_{n\to\infty}\mathfrak{h}(\text{ȽAƚƚ},\hat{\omega}_{n+1})=\mathfrak{h}(\text{ȽAƚƚ},\text{ȽAƚƚ}). \qquad (2.15)$$

On the other hand, by Lemma 2.1.6 and equation (2.3.16), we have

$$\lim_{n\to\infty}\mathfrak{h}(\text{ȽAƚƚ},\hat{\omega}_{n+1})=\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ}). \qquad (2.16)$$

On comparing equation (2.15) and (2.16), we get

$\mathfrak{h}(\text{ȽAƚƚ},\text{ȽAƚƚ})=\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})$.

If possible assume that

$0<\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})$, then from equation (2.1), we get

$$E\left(\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})\right)\le b\ +\ E\left(\alpha(\text{ƚƚ},\text{ƚƚ})\mathfrak{h}(\text{ȽAƚƚ},\text{ȽAƚƚ})\right)$$
$$\le E\left(M(\text{ƚƚ},\text{ƚƚ})\right) \qquad (2.17)$$

Where

$M(\text{ƚƚ},\text{ƚƚ})=\max\{\mathfrak{h}(\text{ƚƚ},\text{ƚƚ}),\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ}),\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})\}$.

If possible, suppose that

$\mathfrak{h}(\text{ƚƚ},\text{ƚƚ})<\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})$.

Then, equation (2.17) implies that

$E\left(\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})\right)<E\left(\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})\right)$.

Which is a contradiction.

Thus,

$0=\mathfrak{h}(\text{ƚƚ},\text{ƚƚ})>\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})$.

Which implies that

$\mathfrak{h}(\text{ƚƚ},\text{ȽAƚƚ})=0$.

Therefore, ȽA has a fixed point ƚƚ in Ꞷ.

Now, orbitally continuous on S Let ĦA is ; then $ῶ_{l+1} = ĦAῶ_l = ĦA(ĦAῶ_0) \to Qŧŧ$ as $l \to \infty$.
By the property of completeness, we say that
$ĦAŧŧ = ŧŧ$.
Therefore, $\mathrm{Fix}(ĦA) \neq \emptyset$.
So, we assume that ĦA is α^* −admissible; this will lead that $\forall\, ŧŧ, ŧŧ^* \in Fix(ĦA)$, we will get $\alpha(ŧŧ, ŧŧ^*) \geq 1$. Therefore, $\mathfrak{h}(ĦAŧŧ, ĦAŧŧ^*) = \mathfrak{h}(ŧŧ, ŧŧ^*) > 0$. From (2.1), we obtain

$$\begin{aligned} ɖ(\mathfrak{h}(ŧŧ, ŧŧ^*)) &= ɖ(\mathfrak{h}(ĦAŧŧ, ĦAŧŧ^*)) \\ &= ɖ(\alpha(ŧŧ, ŧŧ^*)\mathfrak{h}(ĦAŧŧ, ĦAŧŧ^*)) \\ &\leq ɖ(M(ŧŧ, ŧŧ^*)) - b. \end{aligned}$$

Where

$$\begin{aligned} M(ŧŧ, ŧŧ^*) &= max\{\mathfrak{h}(ŧŧ, ŧŧ^*), \mathfrak{h}(ŧŧ, ĦAŧŧ), \mathfrak{h}(ŧŧ^*, ĦAŧŧ^*)\} \\ &= max\{0,0, \mathfrak{h}(ŧŧ, ŧŧ^*)\} \end{aligned}$$

Thus, above becomes

$$ɖ(\mathfrak{h}(ŧŧ, ŧŧ^*)) \leq ɖ(\mathfrak{h}(ŧŧ, ŧŧ^*)) - b$$

Since $b > 0$, and using $(ɖ_1)$, we have

$$\mathfrak{h}(ŧŧ, ŧŧ^*) < \mathfrak{h}(ŧŧ, ŧŧ^*).$$

Which contradicts by our assumption.
Therefore ĦA has a unique fixed point in ω.

Definition 2.4. A self-map ĦA on metric-like space $(ω, \mathfrak{h})$ is called a generalized $\alpha - E$ −Suzuki contraction with condition $b > 0$ exists such that $\forall\, x, y \in ω$ with $ĦAῶ \neq ĦAy$

$$\frac{1}{2}\mathfrak{h}(ῶ, ĦAῶ) < \mathfrak{h}(ῶ, y) \Longrightarrow b + E(\alpha(ῶ, y)\mathfrak{h}(ĦAῶ, ĦAy)) \leq E(M(ῶ, y)) \qquad (2.18)$$

Where $M(x, y) = \{\mathfrak{h}(ῶ, y), \mathfrak{h}(ῶ, ĦAx), \mathfrak{h}(y, ĦAy)\}, \forall\, ῶ, y \in ω$ and $E \in F$.

Theorem 2.5. Let $(ω, \mathfrak{h})$ be a complete metric-like space and $ĦA: ω \to ω$ be a generalized $\alpha - E$ −Suzuki contraction; the subsequent assertion hold:

1. ĦA is $\alpha -$ admissible;
2. $\exists\, ῶ_0 \in ω$ such that $\alpha(ῶ_0, ĦAῶ_0) \geq 1$;
3. $\alpha(ῶ, ῶ) \geq 1$, for all $ῶ \in ω$;
4. ĦA is continuous or orbitally continuous on ω.

Then, the point $ŧŧ \in ω$ is a fixed point of ĦA.
Moreover, if ĦA is α^* −admissible, then the point $ŧŧ \in ω$ is a unique fixed point of ĦA. Furthermore, for every $ῶ_0 \in ω$ if $ῶ_{l+1} = ĦA^{l+1}x_0 \neq ĦAῶ_l, \forall l \geq 0$, then $\lim_{n\to\infty} ĦA^l ῶ_0 = ŧŧ$.

Proof: Let $ῶ_0 \in ω$ be such that $\alpha(ĦAῶ_0, ῶ_0) \geq 1$ and construct a sequence $\{ῶ_l\}$ by $ῶ_{l+1} = ĦA^{l+1}ῶ_0 = ĦAῶ_l, \forall\, l \geq 0$.
If $ῶ_{l_0} = ῶ_{l_0+1}$, it follows that $ĦAῶ_{l_0} = ῶ_{l_0}$ for some $l_0 \geq 0$; then $ῶ_{l_0}$ in ω is a fixed point of ĦA.
Then, Let $ῶ_l \neq ῶ_{l+1} \forall\, l \geq 0$.
So, $\mathfrak{h}(ῶ_l, ῶ_{l+1}) > 0\ \forall\, l \geq 0$.
Using Lemma 2.2, we will have

$$\alpha(ῶ_l, ῶ_{l+1}) > 1 \quad \forall \quad l \geq 0.$$

Then,

$$\alpha(ῶ_l, ῶ_{l+1}) = \alpha(ĦA^l ῶ_0, ĦA^{l+1}ῶ_0) \geq 1\ \forall l \geq 0.$$

According to $ɖ_1$ and Definition 2.4,

$$\begin{aligned} \frac{1}{2}\mathfrak{h}(ῶ_l, ĦAῶ_l) &= \frac{1}{2}\mathfrak{h}(ῶ_l, ῶ_{l+1}) \\ &< \mathfrak{h}(ῶ_l, ῶ_{l+1}). \end{aligned}$$

Now, by equation (2.18), we get

$$\begin{aligned} E(\mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)) &= E(\alpha(ῶ_l, ĦAῶ_l)\mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)) \\ &\leq E(M(ῶ_l, ĦAῶ_l)) - b \end{aligned} \qquad (2.19)$$

Where

$$M(ῶ_l, ĦAῶ_l) = \{\mathfrak{h}(ῶ_l, ĦAῶ_l), \mathfrak{h}(ῶ_l, ĦAῶ_l), \mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)\}$$

If possible, assume that

$$\mathfrak{h}(ῶ_l, ĦAῶ_l) < \mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)$$

Then, equation (2.19), implies that

$$E(\mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)) < E(\mathfrak{h}(ĦAῶ_l, ĦA^2ῶ_l)).$$

A contradiction.

Thus, we get

$E\big(\mathfrak{h}(\mathcal{A}\hat{\varpi}_l, \mathcal{A}^2\hat{\varpi}_l)\big) \le E\big(\mathfrak{h}(\hat{\varpi}_l, \mathcal{A}\hat{\varpi}_l)\big) - b.$

Repeating this process, we get

$$\begin{aligned}
E\big(\mathfrak{h}(\hat{\varpi}_l, \mathcal{A}\hat{\varpi}_l)\big) &= E\big(\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-1}, \mathcal{A}\hat{\varpi}_l)\big)\\
&= E\big(\alpha(\hat{\varpi}_{l-1}, \hat{\varpi}_l)\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-1}, \mathcal{A}\hat{\varpi}_l)\big)\\
&\le E\big(\mathfrak{h}(\hat{\varpi}_{l-1}, \mathcal{A}\hat{\varpi}_{l-1})\big) - b\\
&= E\big(\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-2}, \mathcal{A}\hat{\varpi}_{l-1})\big) - b\\
&= E\big(\alpha(\hat{\varpi}_{l-2}, \hat{\varpi}_{l-1})\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-2}, \mathcal{A}\hat{\varpi}_{l-1})\big) - b\\
&\le E\big(\mathfrak{h}(\hat{\varpi}_{l-2}, \hat{\varpi}_{l-1})\big) - 2b\\
&= E\big(\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-3}, \mathcal{A}\hat{\varpi}_{l-2})\big) - 2b\\
&= E\big(\alpha(\hat{\varpi}_{l-3}, \hat{\varpi}_{l-2})\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-3}, \mathcal{A}\hat{\varpi}_{l-2})\big) - 2b\\
&\le E\big(\mathfrak{h}(\hat{\varpi}_{l-3}, \hat{\varpi}_{l-2})\big) - 3b\\
&\ \ \vdots\\
&\le E\big(\mathfrak{h}(\hat{\varpi}_0, \hat{\varpi}_1)\big) - lb. \qquad (2.20)
\end{aligned}$$

Using the limit on both sides, we get

$$\lim_{n\to\infty} E\big(\mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-1}, \mathcal{A}\hat{\varpi}_l)\big) = -\infty. \qquad (2.21)$$

So, by the definition ($\mathfrak{d}_2$) of definition 1.3 with equation (2.21), we get

$$\lim_{l\to\infty} \mathfrak{h}(\mathcal{A}\hat{\varpi}_{l-1}, \mathcal{A}\hat{\varpi}_l) = 0. \qquad (2.22)$$

By ($\mathfrak{d}_3$) of Definition 1.3, $\exists k$ in $(0,1)$ such that

$$\lim_{l\to\infty}\big(\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})\big)^k E\big(\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})\big) = 0. \qquad (2.23)$$

Also, by equation (2.20), we have

$$[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k\big[E\big(\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})\big) - \mathfrak{d}\big(\mathfrak{h}(\hat{\varpi}_0, \hat{\varpi}_1)\big)\big] \le -[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k lb \le 0. \qquad (2.24)$$

By applying $l \to \infty$ in the previous equation together with the two different equations (2.21) and (2.22), we get

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k = 0. \qquad (2.25)$$

Then, we will discuss two different cases.

Case(i): Consider l is a multiple of 2; by Equation (2.25), we will have

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k = 0. \qquad (2.26)$$

Case(ii): Consider l is not an multiple of 2; by Equation (2.25), we will have

$$\lim_{l\to\infty}(l-1)[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k = 0. \qquad (2.27)$$

By Equations (2.22) and (2.27), we have

$$\lim_{l\to\infty} l[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k = 0. \qquad (2.28)$$

From the previous we will see that $\exists l_1 \in N$ s.t.

$$l[\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1})]^k \le 1\ \forall\, l \ge l_1.$$

Therefore, we have

$\mathfrak{h}(\hat{\varpi}_l, \hat{\varpi}_{l+1}) \le \frac{1}{l^{\frac{1}{k}}}\ \ \forall\, l \ge l_1.$

Then, we will prove that sequence $\{\hat{\varpi}_l\}$ is a Cauchy. Now, $\forall\, p > r \ge l_1$, we have

$$\begin{aligned}
\mathfrak{h}(\hat{\varpi}_p, \hat{\varpi}_r) &\le \mathfrak{h}(\hat{\varpi}_p, \hat{\varpi}_{p-1}) + \mathfrak{h}(\hat{\varpi}_{p-1}, \hat{\varpi}_{p-2}) + \mathfrak{h}(\hat{\varpi}_{p-2}, \hat{\varpi}_{p-3}) + \cdots + \mathfrak{h}(\hat{\varpi}_{r+1}, \hat{\varpi}_r)\\
&< \textstyle\sum_{n=r}^{\infty} \mathfrak{h}(\hat{\varpi}_n, \hat{\varpi}_{n+1})\\
&\le \textstyle\sum_{n=r}^{\infty} \frac{1}{n^{\frac{1}{k}}}.
\end{aligned}$$

Taking limit as $r \to \infty$, we get $\lim_{p,r\to\infty} \mathfrak{h}(\hat{\varpi}_p, \hat{\varpi}_r) = 0$, Since $\le \sum_{n=r}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent if $k < 1$. This proves that sequence $\{\hat{\varpi}_l\}$ in ω is a Cauchy.

So ω is complete, we have $\mathfrak{u} \in \omega$ s.t.

$$\lim_{l\to\infty}\mathfrak{h}(\mathfrak{u}, \hat{\varpi}_l) = \mathfrak{h}(\mathfrak{u}, \mathfrak{u}) = \lim_{m,l\to\infty} \mathfrak{h}(\hat{\varpi}_m, \hat{\varpi}_l) = 0. \qquad (2.29)$$

Next, prove that $\mathfrak{u}$ is a fixed point of $\mathcal{A}$. Now we can claim that

$$\tfrac{1}{2}\mathfrak{h}(\hat{\varpi}_l, \mathcal{A}\hat{\varpi}_l) < \mathfrak{h}(\hat{\varpi}_l, \mathfrak{u})\ or\ \tfrac{1}{2}\mathfrak{h}(\mathcal{A}\hat{\varpi}_l, \mathcal{A}^2\hat{\varpi}_l) < \mathfrak{h}(\mathcal{A}\hat{\varpi}_l, \mathfrak{u}),\ \ l \in \mathbb{N}. \qquad (2.30)$$

Again, suppose that $\exists\, m \in \mathbb{N}$ such that

$\frac{1}{2}ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) < ђ(\tilde{\omega}_m, Ⅱ)$ and $\frac{1}{2}ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m) < ђ(ꜲA\tilde{\omega}_m, Ⅱ)$. (2.31)

Therefore,

$2ђ(\tilde{\omega}_m, Ⅱ) \leq ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) \leq ђ(\tilde{\omega}_m, Ⅱ) + ђ(Ⅱ, ꜲA\tilde{\omega}_m)$,

Which implies that

$ђ(\tilde{\omega}_m, Ⅱ) \leq ђ(Ⅱ, ꜲA\tilde{\omega}_m)$ (2.32)

It follows from the equations (2.31) and (2.32) that

$ђ(\tilde{\omega}_m, Ⅱ) \leq ђ(Ⅱ, ꜲA\tilde{\omega}_m) \leq \frac{1}{2}ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)$ (2.33)

Since $\frac{1}{2}ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) < ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)$, by the condition of the Theorem, we get

$b + E\big(\alpha(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)\big) \leq E\big(M(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)\big)$ (2.34)

Where

$M(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) = max\{ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m), ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m), ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)\}$.

So, if possible, assume that

$ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) < ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)$, then equation (2.34), becomes

$E\big(ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)\big) < E(ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m))$

A contradiction.

Thus,

$b + E\big(\alpha(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)\big) \leq E\big(ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)\big)$. (2.35)

Since $b > 0$, it follows that

$E\big(\alpha(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m)\big) < E\big(ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)\big)$.

So, by ($ɖ_1$), we get

$ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m) < ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m)$. (2.36)

It follows from the equations (2.34), (2.35) and (2.36) that

$$\begin{aligned} ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m) &< ђ(\tilde{\omega}_m, ꜲA\tilde{\omega}_m) \\ &\leq ђ(\tilde{\omega}_m, Ⅱ) + ђ(Ⅱ, ꜲA\tilde{\omega}_m) \\ &\leq \frac{1}{2}ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m) + \frac{1}{2}ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m) \\ &= ђ(ꜲA\tilde{\omega}_m, ꜲA^2\tilde{\omega}_m). \end{aligned} \quad (2.37)$$

Which fails our assumption.

So, Equation (2.30) proves.

Then, using Equation (2.30), for every $l \in \mathbb{N}$

$b + E\big(\alpha(\tilde{\omega}_l, Ⅱ)ђ(ꜲA\tilde{\omega}_l, ꜲAⅡ)\big) < E\big(ђ(\tilde{\omega}_l, Ⅱ)\big)$,

Or

$b + E\big(\alpha(\tilde{\omega}_l, Ⅱ)ђ(ꜲA^2\tilde{\omega}_l, ꜲAⅡ)\big) \leq ɖ\big(ђ(ꜲA\tilde{\omega}_l, Ⅱ)\big) = E\big(ђ(\tilde{\omega}_{l+1}, Ⅱ)\big)$

holds.

In the case (i), from the previous equation (2.30), by the given condition ($ɖ_2$) of definition 1.3, we have

$\lim_{l\to\infty} ђ(ꜲA\tilde{\omega}_l, ꜲAⅡ) = -\infty$.

This will simplify by using the condition ($ɖ_2$) of Definition 1.2 we have

$\lim_{l\to\infty} ђ(ꜲA\tilde{\omega}_l, ꜲAⅡ) = 0$.

Therefore

$ђ(Ⅱ, QⅡ) = \lim_{l\to\infty} ђ(\tilde{\omega}_{l+1}, QⅡ) = \lim_{l\to\infty} ђ(Q\tilde{\omega}_l, QⅡ) = 0$.

In the case (ii), from the previous equation (2.30), by the given condition ($ɖ_2$) of Definition 1.3, we have

$\lim_{l\to\infty} E\big(ђ(ꜲA^2\tilde{\omega}_l, ꜲAⅡ)\big) = -\infty$.

This will simplify by using the condition ($ɖ_2$) of Definition 1.3 we have

$\lim_{l\to\infty} ђ(ꜲA\tilde{\omega}_l, ꜲAⅡ) = 0$. Therefore

$ђ(Ⅱ, ꜲAⅡ) = \lim_{l\to\infty} ђ(\tilde{\omega}_{l+2}, ꜲAⅡ) = \lim_{l\to\infty} ђ(ꜲA^2\tilde{\omega}_l, ꜲAⅡ) = 0$.

So, ꜲA has a fixed point Ⅱ in ɷ.

So, we consider ꜲA is orbitally continuous on ɷ; then $\tilde{\omega}_{l+1} = ꜲA\tilde{\omega}_l = ꜲA(ꜲA\tilde{\omega}_0) \to ꜲAⅡ$ as $l \to \infty$.

By using the completeness property of metric space,

we get $ꜲAⅡ = Ⅱ$.

So, Fix$(ꜲA) \neq \emptyset$.

One more time, we consider that ꜲA is α^* −admissible; this implies that $\forall\, Ⅱ, Ⅱ^* \in Fix(ꜲA)$, we have $\alpha(Ⅱ, Ⅱ^*) \geq 1$.

Therefore, $ђ(ꜲAⅡ, ꜲAⅡ^*) = ђ(Ⅱ, Ⅱ^*) > 0$.

So, we have $0 = \frac{1}{2}\text{ђ}(\text{ŧ}, \text{Ⱥŧ}) < \text{ђ}(\text{ŧ}, \text{ŧ}^*)$, and by given assertion of the previous Theorem, we got

$E\big(\text{ђ}(\text{ŧ}, \text{ŧ}^*)\big) = \text{ɖ}\big(\text{ђ}(\text{Ⱥŧ}, \text{Ⱥŧ}^*)\big) = E\big(\alpha(\text{ŧ}, \text{ŧ}^*)\text{ђ}(\text{Ⱥŧ}, \text{Ⱥŧ}^*)\big) \leq \text{Ⱥ}\big(M(\text{ŧ}, \text{ŧ}^*)\big) - b.$

Where

$M(\text{ŧ}, \text{ŧ}^*) = max\{\text{ђ}(\text{ŧ}, \text{ŧ}^*), \text{ђ}(\text{ŧ}, \text{Ⱥŧ}), \text{ђ}(\text{ŧ}^*, \text{Ⱥŧ}^*)\}$

$\quad = max\{0,0, \text{ђ}(\text{ŧ}, \text{ŧ}^*)\}$

Thus, above becomes

$\text{ɖ}\big(\text{ђ}(\text{ŧ}, \text{ŧ}^*)\big) \leq \text{ɖ}\big(\text{ђ}(\text{ŧ}, \text{ŧ}^*)\big) - b$

Since $b > 0$, and using (ɖ_1), we have

$\text{ђ}(\text{ŧ}, \text{ŧ}^*) < \text{ђ}(\text{ŧ}, \text{ŧ}^*)$.

Which fails by our assumption.

Therefore Ⱥ has a unique fixed point in ɷ.

CONFLICT OF INTEREST

There is no conflict of interest between the authors.

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