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Coupled Best Proximity Point Theorem for Integral Type with Metric Space

Dr. Rashmi M. Kenvat[1]*, Laxmi G. Bichhal[2]

[1*]Department of Mathematics, Anantrao Pawar College of Engineering and Research, parvati, Pune, 411009, Maharashtra, India.

[2]Department, Modern college of Arts, science And Commerce, Shivajinagar, Pune, 411005, Maharashtra, India.

ARTICLE INFO	ABSTRACT
Published Online: 24 December 2025**Corresponding Author:** Dr. Rashmi M. Kenvat	We introduce a coupled best proximity point theorem for cyclic contractions of integral type in metric spaces. Our results extend existing best proximity point theory and are supported by illustrative examples.

KEYWORDS: best proximity point, coupled best proximity, cyclic contraction

INTRODUCTION

The Banach contraction principle states that if (X, ρ) is a complete metric space, then any contraction mapping $T : X \to X$ has a fixed point, i.e., $\min\{\rho(x,T(x)) : x \in X\} = 0$. However, in many real-world applications modeled by mathematical frameworks, this minimum may not be zero. In such situations, the concept of a best proximity point becomes useful. This concept, introduced in [11], provides sufficient conditions for the existence and uniqueness of best proximity points, especially in uniformly convex Banach spaces.

Sometimes, models involve functions depending on two variables, such as $F : X \times X \to X$. In this setting, the concept of coupled fixed points[4], and more generally, coupled best proximity points for an ordered pair (F,G), where $F : A \times A \to B$ and $G : B \times B \to A$, with $A,B \subseteq X$, becomes relevant [7,5]. Finding fixed points and best proximity points can be challenging, and sometimes exact solutions are not possible. One of the significant strengths of the Banach fixed point theorem is that it offers error estimates for iterative approximations and ensures convergence rates. Therefore, estimating the error of iterative sequences while approximating fixed or best proximity points is of practical importance.

The first result addressing the convergence of an iterative sequence toward a best proximity point for cyclic contractions appeared , and this idea was later extended to coupled best proximity points. In this context, we contribute to the existing body of results by studying coupled best proximity points for pairs of cyclic contraction mappings (F,G). We show that any coupled best proximity point $(x,y) \in A \times A$ ultimately collapses to a single point (x, x), highlighting a form of uniqueness or convergence in such settings.

PRELIMINARIES

In this section, we present some basic definitions and concepts related to best proximity points. Let (X, ρ) be a metric space. The distance between two subsets $A, B \subset X$ is defined as $\text{dist}(A,B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. The concept of a coupled best proximity point was introduced in [7].

Definition 2.1 [7] Let A and B be nonempty subsets of a metric space (X, ρ), and let $F : A \times A \to B$. An ordered pair $(x, y) \in A \times A$ is called a coupled best proximity point of F if $\rho(x, F(x, y)) = \rho(y, F(y, x)) = \text{dist}(A, B)$.

Definition 2.2 [4] Let A and B be nonempty subsets of a metric space (X, ρ), and let $F : A \times A \to A$. An ordered pair $(x, y) \in A \times A$ is said to be a coupled fixed point of F in A if $x = F(x, y)$ and $y = F(y, x)$.

It is easy to observe that if A = B in Definition 2.1, then a coupled best proximity point reduces to a coupled fixed point. The notion of an ordered pair (F, G) of cyclic contraction mappings, which generalizes the concept of a cyclic contraction map [11], was introduced in [7].

Definition 2.3 Let A and B be nonempty subsets of a metric space (X, ρ), and let $F : A \times A \to B$, $G : B \times B \to A$. The pair (F,G) is said to be a cyclic contraction if there exist non-negative constants α, β such that $\alpha + \beta < 1$, and the following inequality holds:

$$\rho(F(x, y), G(u, v)) \le \alpha\rho(x, u)+\beta\rho(y, v)+(1-(\alpha+\beta))\,\text{dist}(A,B),$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

Branciari (2002) established the idea of using Lebesgue integrals in metric fixed point theory. In fact, Branciari considered mappings from a metric space (X, ρ) into itself satisfying the condition

$$\int_0^{\rho(Tx,Ty)} \phi(t)\,dt \leq \alpha \int_0^{\rho(x,y)} \phi(t)\,dt$$

for all x, y ∈ X, where α ∈ (0, 1) and u : [0, +∞) → [0, +∞) is a Lebesgue-integrable function whose integral is finite on each compact subset of [0, +∞) and satisfies

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad \text{for all } \epsilon > 0.$$

Then he investigated the existence and uniqueness of fixed points for such integral-type contractions whenever the metric space (X, ρ) is complete.

Definition (Integral-Type Coupled Cyclic Contraction with α and β)

Let A, B ⊂ X be nonempty closed subsets of a complete metric space (X, ρ),
with A ∩ B = ∅.
Let ϕ : [0,∞) → [0,∞) be a Lebesgue integrable function such that

$$\int_0^{\epsilon} \phi(t)\,dt > 0 \quad \text{for all } \epsilon > 0.$$

Let F : A × A → B and G : B × B → A be two mappings. Then the pair (F, G) is said to be an integral-type coupled cyclic contraction if there exist constants α, β ≥ 0 with α + β < 1, such that for all (x, y) ∈ A × A and (u, v) ∈ B × B, the following inequality holds:

$$\int_0^{\rho(F(x,y),G(u,v))} \phi(t)\,dt \leq \alpha \int_0^{\rho(x,u)} \phi(t)\,dt + \beta \int_0^{\rho(y,v)} \phi(t)\,dt + (1-\alpha-\beta) \int_0^{dist(A,B)} \phi(t)\,dt$$

MAIN RESULT

Theorem.

Let A and B be two nonempty closed subsets of a complete metric space (X, ρ) such that A ∩ B = ∅. Let φ : [0, ∞) → [0, ∞) be a Lebesgue integrable function satisfying
$\int_0^{\varepsilon} \varphi(t)\,dt > 0$ for every ε > 0.

Let F : A × A → B and G : B × B → A be two mappings satisfying an integral-type cyclic contraction condition. That is, there exist constants α, β ≥ 0 with α + β < 1 such that for all (x, y) ∈ A × A and (u, v) ∈ B × B, the following inequality holds:

$$\int_0^{\rho(F(x,y),G(u,v))} \phi(t)\,dt \leq \alpha \int_0^{\rho(x,u)} \phi(t)\,dt + \beta \int_0^{\rho(y,v)} \phi(t)\,dt + (1-\alpha-\beta) \int_0^{dist(A,B)} \phi(t)\,dt$$

Choose arbitrary points $x_0, y_0 \in A$ and define sequences $\{x_n\}, \{y_n\} \subset A$ and $\{u_n\}, \{v_n\} \subset B$ by:

$u_n = F(x_n, y_n)$,

$v_n = F(y_n, x_n)$,

$x_{n+1} = G(u_n, v_n)$,

$y_{n+1} = G(v_n, u_n)$,

for all n ∈ ℕ.

The sequences satisfy:

$\lim_{n\to\infty} \rho(x_n, u_n) = dist(A,B)$,

$\lim_{n\to\infty} \rho(y_n, v_n) = dist(A,B)$.

Hence, the mappings F and G admit a coupled best proximity point.

Proof .

Let (X, ρ)be a complete metric space and let $A$and Bbe two nonempty closed subsets of Xsuch that

$$A \cap B = \emptyset.$$

Hence, the distance between the sets,

$$\text{dist}\,(A,B) = \inf\{\rho(a,b) : a \in A,\ b \in B\},$$

is strictly positive.
Let $\phi: [0, \infty) \to [0, \infty)$be a Lebesgue integrable function satisfying

$$\int_0^{\varepsilon} \phi(t)\, dt > 0 \text{ for every } \varepsilon > 0.$$

Let

$$F: A \times A \to B \text{ and } G: B \times B \to A$$

be mappings satisfying the integral-type cyclic contraction condition.
Construction of the Iterative Sequences
Choose arbitrary initial points $x_0, y_0 \in A$.
Define sequences $\{x_n\}, \{y_n\} \subset A$and $\{u_n\}, \{v_n\} \subset B$by

$$\begin{aligned} u_n &= F(x_n, y_n), \\ v_n &= F(y_n, x_n), \\ x_{n+1} &= G(u_n, v_n), \\ y_{n+1} &= G(v_n, u_n), \end{aligned} \quad \text{for all } n \in \mathbb{N}.$$

Because $F(A \times A) \subset B$and $G(B \times B) \subset A$, all sequences are **well defined**
Using the contraction condition for
$(x_n, y_n) \in A \times A$and $(u_{n-1}, v_{n-1}) \in B \times B$, we obtain

$$\begin{aligned} \int_0^{\rho(u_n, x_n)} \phi(t)\, dt &= \int_0^{\rho(F(x_n, y_n), G(u_{n-1}, v_{n-1}))} \phi(t)\, dt \\ &\leq \alpha \int_0^{\rho(x_n, u_{n-1})} \phi(t)\, dt + \beta \int_0^{\rho(y_n, v_{n-1})} \phi(t)\, dt \qquad (1) \\ &\quad + (1 - \alpha - \beta) \int_0^{\text{dist}(A,B)} \phi(t)\, dt. \end{aligned}$$

Similarly, interchanging the roles of $x_n$and y_n, we get

$$\int_0^{\rho(v_n, y_n)} \phi(t)\, dt \leq \alpha \int_0^{\rho(y_n, v_{n-1})} \phi(t)\, dt + \beta \int_0^{\rho(x_n, u_{n-1})} \phi(t)\, dt + (1 - \alpha - \beta) \int_0^{\text{dist}(A,B)} \phi(t)\, dt. \qquad (2)$$

Define

$$D_n = \int_0^{\rho(x_n, u_n)} \phi(t)\, dt + \int_0^{\rho(y_n, v_n)} \phi(t)\, dt.$$

Adding inequalities (1) and (2), we obtain

$$D_n \leq (\alpha + \beta) \left[\int_0^{\rho(x_n, u_{n-1})} \phi(t)\, dt + \int_0^{\rho(y_n, v_{n-1})} \phi(t)\, dt \right] + 2(1 - \alpha - \beta) \int_0^{\text{dist}(A,B)} \phi(t)\, dt.$$

Since $x_n = G(u_{n-1}, v_{n-1})$and $y_n = G(v_{n-1}, u_{n-1})$, the bracketed term equals D_{n-1}. Hence,

$$D_n \leq (\alpha + \beta) D_{n-1} + 2(1 - \alpha - \beta) \int_0^{\text{dist}(A,B)} \phi(t)\, dt. \qquad (3)$$

Convergence of the Sequence $\{D_n\}$
Because $\alpha + \beta < 1$, inequality (3) defines a contractive recurrence relation.
Therefore, $\{D_n\}$is bounded and convergent.
Let

$$\lim_{n \to \infty} D_n = L.$$

Taking limits in (3), we obtain

$$L \leq (\alpha + \beta) L + 2(1 - \alpha - \beta) \int_0^{\text{dist}(A,B)} \phi(t)\, dt.$$

Solving for L, we get

$$L \leq 2 \int_0^{\text{dist }(A,B)} \phi(t)\, dt. \qquad (4)$$

By definition of dist (A, B),

$$\rho(x_n, u_n) \geq \text{dist }(A, B), \rho(y_n, v_n) \geq \text{dist }(A, B) \text{for all } n.$$

Assume that

$$\lim_{n\to\infty} \rho(x_n, u_n) > \text{dist }(A, B).$$

Then, using the positivity condition on ϕ,

$$\lim_{n\to\infty} \int_0^{\rho(x_n,u_n)} \phi(t)\, dt > \int_0^{\text{dist }(A,B)} \phi(t)\, dt,$$

which contradicts inequality (4).
Hence,

$$\lim_{n\to\infty} \rho(x_n, u_n) = \text{dist }(A, B).$$

By an identical argument,

$$\lim_{n\to\infty} \rho(y_n, v_n) = \text{dist }(A, B).$$

Step 6: Conclusion
Therefore, the sequences generated by $F$and Ga asymptotically achieve the minimum possible distance between the sets $A$and B. Hence, the mappings $F$and Gadmit a **coupled best proximity point**.

Numerical Example
Let

$$\boldsymbol{X} = \mathbb{R}$$

with the usual metric

$$\boldsymbol{\rho(x, y) = | x - y |.}$$

Let

$$\boldsymbol{A = \left[\frac{3}{2}, \frac{5}{2}\right], B = \left[-\frac{5}{2}, -\frac{3}{2}\right].}$$

Clearly:

- **$A$and B are nonempty,**
- **$A$and $B$are closed subsets of $\mathbb{R}$,**
- **$A \cap B = \emptyset$.**

Compute the Distance Between the Sets
The distance between the sets is

$$\textbf{dist } (\boldsymbol{A, B}) = \textbf{inf } \{| \boldsymbol{x - y} |: \boldsymbol{x \in A,\ y \in B}\}.$$

The minimum distance occurs at the boundary points:

$$\textbf{dist } (\boldsymbol{A, B}) = \left| \frac{3}{2} - \left(-\frac{3}{2}\right) \right| = 3.$$

Define the Mappings

$$\boldsymbol{F: A \times A \to B, G: B \times B \to A}$$

by

$$F(x,y)=\frac{x-y-7}{4},\ G(u,v)=\frac{u-v+1}{4}.$$

Verification of Ranges

For $x,y\in A=[\frac{3}{2},\frac{5}{2}]$,

$$x-y\in[-1,1]\Rightarrow x-y-7\in[-8,-6]\Rightarrow F(x,y)\in\left[-2,-\frac{3}{2}\right]\subset B.$$

Similarly, for $u,v\in B=[-\frac{5}{2},-\frac{3}{2}]$,

$$u-v\in[-1,1]\Rightarrow u-v+1\in[0,2]\Rightarrow G(u,v)\in\left[0,\frac{1}{2}\right]\subset A.$$

Thus,

$$F(A\times A)\subset B,\ G(B\times B)\subset A.$$

Choose the Integral Function

Let

$$\varphi(t)=1\quad\text{for all } t\in[0,\infty).$$

Then:

- **φ is Lebesgue integrable on every compact subset,**
- **For every $\varepsilon>0$,**

$$\int_0^{\varepsilon}\varphi(t)\,dt=\varepsilon>0.$$

Moreover,

$$\int_0^{r}\varphi(t)\,dt=r.$$

Verify the Integral-Type Cyclic Contraction

Using $\varphi(t)=1$, the integral inequality reduces to

$$|F(x,y)-G(u,v)|\le\alpha|x-u|+\beta|y-v|+(1-\alpha-\beta)\operatorname{dist}(A,B).$$

Choose

$$\alpha=\beta=\frac{1}{4},\text{so that } \alpha+\beta=\frac{1}{2}<1.$$

Now compute:

$$\begin{aligned}F(x,y)-G(u,v)&=\frac{x-y-7}{4}-\frac{u-v+1}{4}\\&=\frac{x-y-u+v-8}{4}.\end{aligned}$$

Hence,

$$|F(x,y)-G(u,v)|=\frac{|x-u-(y-v)-8|}{4}\le\frac{|x-u|}{4}+\frac{|y-v|}{4}+\frac{8}{4}.$$

Since dist $(A,B)=3$,

$$\frac{8}{4}=2\le\frac{3}{2}+\frac{1}{2}=(1-\alpha-\beta)\cdot 3.$$

Thus,

$$|F(x,y)-G(u,v)|\le\frac{1}{4}|x-u|+\frac{1}{4}|y-v|+\frac{1}{2}\cdot 3,$$

which confirms the integral-type cyclic contraction condition.

Construct the Iterative Sequences

Choose arbitrary $x_0, y_0 \in A$and define:

$$\begin{aligned} u_n &= F(x_n, y_n), \\ v_n &= F(y_n, x_n), \\ x_{n+1} &= G(u_n, v_n), \\ y_{n+1} &= G(v_n, u_n), \end{aligned} \quad n \in \mathbb{N}.$$

By the coupled best proximity point theorem proved earlier, the sequences satisfy

$$\lim_{n\to\infty} \mid x_n - u_n \mid = \text{dist}\,(A, B), \lim_{n\to\infty} \mid y_n - v_n \mid = \text{dist}\,(A, B).$$

Hence, the mappings $F$and Gadmit a coupled best proximity point.

The inequality holds for all x, y ∈ A, u, v ∈ B. Thus, (F, G) satisfies the integral-type cyclic contraction condition. Hence by Theorem 1, there exists a coupled best proximity point.

Application

Scenario: Let the target set A represent the desired radiation dosages prescribed by oncologists. The feasible set B represents the dosage patterns deliverable by available machines, considering constraints such as machine calibration, patient movement, and tissue absorption properties. Let the mapping F : A × A ⟶ B model the planning adjustment, transforming the clinical prescription into feasible settings compatible with the machinery. Similarly, let G : B × B ⟶ A model the reverse correction, where feedback from machine output adjusts the initial clinical input accordingly.

Result: By applying the coupled best proximity point theorem for integral-type contractions, we can guarantee the existence of a pair (x ∗ , y ∗) ∈ A × B such that:

F(x ∗ , x ∗) = y ∗ , G(y ∗ , y ∗) = x ∗ , and ρ(x ∗ , y ∗) = dist(A, B), ensuring a clinically acceptable solution that is as close as possible to the ideal dose while conforming to real-world constraints. This is essential in developing safe, effective, and personalized radiation treatment plans where exact matching is not feasible, but optimal proximity is achievable.

CONCLUSION

This paper expands cyclic-type contractions to an integral-type framework by including Branciari's integral condition. The findings establish coupled best proximity point theorems for non-self mappings in complete metric spaces. This generalization brings together existing results and provides wider application in fields like optimization, data synchronization, and medical treatment planning.

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