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ON SOME NEW MIXED MODULAR EQUATIONS OF COMPOSITE DEGREES

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Abstract. On pages 242-250 of his second notebook, Ramanujan records modular equations of composite degrees. All these have been proved by B. C. Berndt using either the method of parametrization or the theory of modular forms. In this paper, we establish several new U - V modular equations of composite degrees using eta-function identities by the method familiar to Ramanujan. We also establish several new general formulas to compute Ramanujan - Weber class invariants g_n and G_n.

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1. Introduction

Ramanujan's general theta-function [10] $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1, \tag{1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{6}$$

where $0<k<1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\, n!}z^n, \quad 0\le |z|<1,$$

where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad \text{for } \ n \text{ is a positive integer}$$

and a, b and c are complex numbers such that $c \neq 0, -1, -2, \ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{7}$$

holds for some positive integer n. Then, modular equation of degree n is a relation between the moduli k and l which is induced by (7). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{8}$$

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2} \ \text{ and } \ n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{9}$$

holds. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (9). We say that β, γ and δ are of degrees n_1, n_2 and n_3, respectively over α. The multipliers m and m' are associated with α, β γ and δ.

M. S. Mahadeva Naika, S. Chandankumar and B. Hemanthkumar [7] established some new mixed modular equations relating A_1 with A_r, where

$$A_r = \frac{f^2(-q^{3r})}{q^{\frac{r}{6}} f(-q^r) f(-q^{9r})}$$

and $r \in \{2, 3, 5, 7, 11, 13\}$. They also established several explicit evaluations of cubic singular moduli.

In Chapter 25 of his Second Notebook, Ramanujan states twenty three beautiful P - Q eta-function identities or P - Q modular equations. These are identities involving quotients of eta-functions, which are designated as P or Q by Ramanujan. Elementary proofs of eighteen of these twenty three P - Q identities by employing the theory of theta-functions in the Spirit of Ramanujan and remaining five using theory of modular forms can be found in [3]. For more details, one can see, [6], [8] and [9].

In section 2, we establish several new U - V mixed modular equations of degrees. In section 3, we establish several general formula to compute Ramanujan-Weber class invariants g_n and G_n. by using mixed modular equations obtained in section 2.

2. New $U - V$ mixed modular equations

In this section, we establish some new $U - V$ mixed modular equations of composites degrees using Ramanujan's eta-function identities by elementary method.

Theorem 2.1. *If* $U = \frac{f^2(-q^2) f^2(-q^3)}{q^{1/6} f^2(-q) f^2(-q^6)}$ *and* $V = \frac{f^2(-q^4) f^2(-q^6)}{q^{1/3} f^2(-q^2) f^2(-q^{12})}$, *then*

$$U^2V^2 + \frac{1}{U^2V^2} - 6 = \left(UV + \frac{1}{UV}\right)\left(\frac{U^3}{V^3} + \frac{V^3}{U^3}\right). \tag{10}$$

Proof. From the page 327 of Chapter 25 of Ramanujan's second notebook [2, Entry 51, p.204], [10, Entry 51, p.327], we have

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3, \tag{11}$$

where $P = \dfrac{f^2(-q)}{q^{1/6}f^2(-q^3)}$ and $Q = \dfrac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}$.
The equation(11) can be re-written as

$$U^2Q^2 + \frac{9U^4}{Q^2} - \left(U^6+1\right) = 0. \tag{12}$$

Replacing q by q^2 in the above equation (11), we find that

$$QR + \frac{9}{QR} = \left(\frac{R}{Q}\right)^3 + \left(\frac{Q}{R}\right)^3, \tag{13}$$

where $R = \frac{f^2(-q^4)}{q^{2/3}f^2(-q^{12})}$.
The equation(13) can be re-written as

$$V^4Q^2 + \frac{9V^2}{Q^2} - \left(V^6+1\right) = 0. \tag{14}$$

Solving the above equations (12) and (14) by cross rule multiplication, we find that

$$\begin{aligned}\frac{Q^2}{9V^2\left(U^6+1\right) - 9U^4\left(V^6+1\right)} &= \frac{1/Q^2}{U^2\left(V^6+1\right) - V^4\left(U^6+1\right)} \\ &= \frac{1}{9U^2V^2 - 9U^4V^4},\end{aligned} \tag{15}$$

which implies

$$Q^2 = \frac{V^2\left(U^6+1\right) - U^4\left(V^6+1\right)}{U^2V^2 - U^4V^4} \tag{16}$$

and

$$\frac{1}{Q^2} = \frac{U^2\left(V^6+1\right) - V^4\left(U^6+1\right)}{9\left(U^2V^2 - U^4V^4\right)}. \tag{17}$$

Multiplying the equations (16) and (17), we obtain

$$\begin{aligned}&(UV-1)^2(UV+1)^2 \\ &\times\left(U^6+V^6+U^8V^2+U^2V^8-U^6V^6+6U^4V^4-U^2V^2\right) = 0.\end{aligned} \tag{18}$$

As $q \to 0$, $UV - 1 \neq 0$ and $UV + 1 \neq 0$. Hence, we obtain (10). □

Theorem 2.2. *If* $U = \dfrac{f^2(-q)f^2(-q^2)}{q^{1/2}f^2(-q^3)f^2(-q^6)}$ *and* $V = \dfrac{f^2(-q^2)f^2(-q^4)}{qf^2(-q^6)f^2(-q^{12})}$, *then*

$$\frac{U^4}{V^4} + \frac{V^4}{U^4} = 7\left(\frac{U^2}{V^2} + \frac{V^2}{U^2}\right) + \left(UV + \frac{81}{UV}\right)\left(\frac{U}{V} + \frac{V}{U}\right) + 24. \tag{19}$$

Proof. The equations (11) and (13) can be written respectively as

$$Q^6+\frac{U^6}{Q^6}-(U^4+9U^2)=0 \tag{20}$$

and

$$Q^6+\frac{V^6}{Q^6}-(V^4+9V^2)=0, \tag{21}$$

where U and V are defined as in the Theorem (2.2).

Solving the above equations (20) and (21) by cross rule multiplication, we find that

$$Q^6=\frac{V^6\left(U^4+9U^2\right)-U^6\left(V^4+9V^2\right)}{V^6-U^6} \tag{22}$$

and

$$\frac{1}{Q^6}=\frac{V^4+9V^2-U^4-9U^2}{V^6-U^6}. \tag{23}$$

Multiplying the equations (22) and (23), we obtain

$$(U+V)^2(U-V)^2\left(-U^6V^4-U^4V^6+U^8-7U^6V^2-24U^4V^4\right.$$
$$\left.-7U^2V^6+V^8-81U^4V^2-81U^2V^4\right)=0. \tag{24}$$

As $q\to 0$, $U-V\neq 0$ and $U+V\neq 0$ in the above equation. Hence, we obtain (19). □

Theorem 2.3. *If* $U=\dfrac{f(-q^2)f(-q^5)}{q^{1/6}f(-q)f(-q^{10})}$ *and* $V=\dfrac{f(-q^4)f(-q^{10})}{q^{1/3}f(-q^2)f(-q^{20})}$, *then*

$$U^2V^2+\frac{1}{U^2V^2}=\left(UV+\frac{1}{UV}\right)\left(\frac{U^3}{V^3}+\frac{V^3}{U^3}\right)+2. \tag{25}$$

Proof. From the page 327 of Chapter 25 of Ramanujan's second notebook [2, Entry 53, p.206], [10, Entry 53, p.325], we have

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3, \tag{26}$$

where $P=\dfrac{f(-q)}{q^{1/6}f(-q^5)}$ and $Q=\dfrac{f(-q^2)}{q^{1/3}f(-q^{10})}$.

Replacing q by q^2 in the above equation (26), we find that

$$QR+\frac{5}{QR}=\left(\frac{R}{Q}\right)^3+\left(\frac{Q}{R}\right)^3, \tag{27}$$

where $R = \dfrac{f(-q^4)}{q^{2/3}f(-q^{20})}$.
The equations (26) and (27) respectively can be written as

$$U^2Q^2 + \frac{5U^4}{Q^2} = U^6 + 1 \tag{28}$$

and

$$V^4Q^2 + \frac{5V^2}{Q^2} = V^6 + 1, \tag{29}$$

where U and V are defined as in the Theoerm (2.3).
Solving the above equations (28) and (29) for Q^2 and $\dfrac{1}{Q^2}$, we find that

$$Q^2 = \frac{V^2(U^6+1) - U^4(V^6+1)}{U^2V^2 - U^4V^4} \tag{30}$$

and

$$\frac{1}{Q^2} = \frac{U^2(V^6+1) - V^4(U^6+1)}{5(U^2V^2 - U^4V^4)}. \tag{31}$$

Multiplying the above equations (30) and (31), we get

$$\begin{aligned} &(UV+1)^2(UV-1)^2 \\ &\times\left(U^6+V^6+U^8V^2+U^2V^8+2U^4V^4-U^6V^6-U^2V^2\right) = 0. \end{aligned} \tag{32}$$

As $q \to 0$, $UV - 1 \neq 0$ and $UV + 1 \neq 0$ in the above equation. Hence, we obtain (25). □

Theorem 2.4. *If* $U = \dfrac{f(-q)f(-q^2)}{q^{1/2}f(-q^5)f(-q^{10})}$ *and* $V = \dfrac{f(-q^2)f(-q^4)}{qf(-q^{10})f(-q^{20})}$, *then*

$$\frac{U^4}{V^4} + \frac{V^4}{U^4} - 3\left(\frac{U^2}{V^2} + \frac{V^2}{U^2}\right) - 12 = \left(UV + \frac{25}{UV}\right)\left(\frac{U}{V} + \frac{V}{U}\right). \tag{33}$$

Proof. The equations (26) and (27) can be written as

$$Q^6 + \frac{U^6}{Q^6} = U^4 + 5U^2 \tag{34}$$

and

$$Q^6 + \frac{V^6}{Q^6} = V^4 + 5V^2, \tag{35}$$

where U and V are defined as in the Theoerm (2.4).
Solving the above equations (34) and (35) for Q^6 and $\dfrac{1}{Q^6}$, we get

$$Q^6 = \frac{V^6(U^4+5U^2) - U^6(V^4+5V^2)}{V^6 - U^6} \tag{36}$$

and

$$\frac{1}{Q^6} = \frac{V^4 + 5V^2 - U^4 - 5U^2}{V^6 - U^6}. \tag{37}$$

Multiplying the above equations (36) and (37), we obtain (33). □

Theorem 2.5. *If* $U = \dfrac{f^2(-q^2)f^2(-q^7)}{q^{1/2}f^2(-q)f^2(-q^{14})}$ *and* $V = \dfrac{f^2(-q^4)f^2(-q^{14})}{qf^2(-q^2)f^2(-q^{28})}$, *then*

$$\begin{aligned} &U^2V^2 + \frac{1}{U^2V^2} + 8\left(\frac{U^2}{V^2} + \frac{V^2}{U^2}\right) + 60 \\ &= \left(UV + \frac{1}{UV}\right)\left[\frac{U^3}{V^3} + \frac{V^3}{U^3} + 8\left(\frac{U}{V} + \frac{V}{U}\right)\right]. \end{aligned} \tag{38}$$

Proof. From the page 327 of Chapter 25 of Ramanujan's second notebook [2, Entry 55, p.209], [10, Entry 55, p.327], we have

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 8\left(\frac{Q}{P} + \frac{P}{Q}\right), \tag{39}$$

where $P = \dfrac{f^2(-q)}{q^{1/2}f^2(-q^7)}$ and $Q = \dfrac{f^2(-q^2)}{qf^2(-q^{14})}$.
Replacing q by q^2 in the equation (39), we find that

$$QR + \frac{7}{QR} = \left(\frac{R}{Q}\right)^3 + \left(\frac{Q}{R}\right)^3 - 8\left(\frac{R}{Q} + \frac{Q}{R}\right), \tag{40}$$

where $R = \dfrac{f^2(-q^4)}{q^2f^2(-q^{28})}$.
The equations (39) and (40) can be written as

$$U^2Q^2 + \frac{7U^4}{Q^2} = U^6 - 8U^4 - 8U^2 + 1 \tag{41}$$

and

$$V^4Q^2 + \frac{7V^2}{Q^2} = V^6 - 8V^4 - 8V^2 + 1, \tag{42}$$

where U and V are defined as in the Theorem (2.5).
Solving the above equations (41) and (42) for Q^2 and $\dfrac{1}{Q^2}$, we get

$$Q^2 = \frac{V^2(U^6 - 8U^4 - 8U^2 + 1) - U^4(V^6 - 8V^4 - 8V^2 + 1)}{U^2V^2 - U^4V^4} \tag{43}$$

and

$$\frac{1}{Q^2} = \frac{U^2(V^6 - 8V^4 - 8V^2 + 1) - V^4(U^6 - 8U^4 - 8U^2 + 1)}{7(U^2V^2 - U^4V^4)}. \tag{44}$$

Using the equations (43) and (44), we obtain (38). □

Theorem 2.6. *If* $U^2 = \dfrac{f(-q^2)f(-q^9)}{q^{1/3}f(-q)f(-q^{18})}$ *and* $V^2 = \dfrac{f(-q^4)f(-q^{18})}{q^{2/3}f(-q^2)f(-q^{36})}$, *then*

(45) $$U^2V^2 + \frac{1}{U^2V^2} = \left(UV + \frac{1}{UV}\right)\left(\frac{U^3}{V^3} + \frac{V^3}{U^3}\right).$$

Proof. From the page 327 of Chapter 25 of Ramanujan's second notebook [2, Entry 56, p.210], [10, Entry 56, p.327], we have

(46) $$\sqrt{\frac{Q^3}{P^3}} + \sqrt{\frac{P^3}{Q^3}} = \sqrt{PQ} + \frac{3}{\sqrt{PQ}},$$

where $P = \dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $Q = \dfrac{f(-q^2)}{q^{2/3}f(-q^{18})}$.
Replacing q by q^2 in the equation (46), we find that

(47) $$\sqrt{\frac{R^3}{Q^3}} + \sqrt{\frac{Q^3}{R^3}} = \sqrt{QR} + \frac{3}{\sqrt{QR}},$$

where $R = \dfrac{f(-q^4)}{q^{4/3}f(-q^{36})}$. □

The equations (46) and (47) can be written as

(48) $$U^2Q + \frac{3U^4}{Q} = U^6 + 1$$

and

(49) $$V^4Q + \frac{3V^2}{Q} = V^6 + 1,$$

where U and V are defined as in the Theorem (2.6).
Solving the above equations (48) and (49) for Q and $\dfrac{1}{Q}$, we get

(50) $$Q = \frac{V^2(U^6+1) - U^4(V^6+1)}{U^2V^2 - U^4V^4}$$

and

(51) $$\frac{1}{Q} = \frac{U^2(V^6+1) - V^4(U^6+1)}{3(U^2V^2 - U^4V^4)}.$$

Using the equations (50) and (51), we obtain (46).

Theorem 2.7. *If* $U^2 = \dfrac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})}$ *and* $V^2 = \dfrac{f(-q^2)f(-q^4)}{q^2f(-q^{18})f(-q^{36})}$, *then*

(52) $$\frac{U^4}{V^4} + \frac{V^4}{U^4} = \frac{U^2}{V^2} + \frac{V^2}{U^2} + \left(UV + \frac{9}{UV}\right)\left(\frac{U}{V} + \frac{V}{U}\right) + 6.$$

Proof. The equations (46) and (47) can be written as

$$Q^3 + \frac{U^6}{Q^3} = U^4 + 3U^2 \tag{53}$$

and

$$Q^3 + \frac{V^6}{Q^3} = V^4 + 3V^2, \tag{54}$$

where U and V are defined as in the Theorem (2.7).
Solving the above equations (53) and (54) for Q^3 and $\frac{1}{Q^3}$, we get

$$Q^3 = \frac{V^6(U^4 + 3U^2) - U^6(V^4 + 3V^2)}{V^6 - U^6} \tag{55}$$

and

$$\frac{1}{Q^3} = \frac{V^4 + 3V^2 - U^4 - 3U^2}{V^6 - U^6}. \tag{56}$$

Using the equations (55) and (56), we obtain (52). □

Theorem 2.8. *If* $U = \frac{f(-q^2)f(-q^{13})}{q^{1/2}f(-q)f(-q^{26})}$ *and* $V = \frac{f(-q^4)f(-q^{26})}{qf(-q^2)f(-q^{52})}$, *then*

$$U^2V^2 + \frac{1}{U^2V^2} + \frac{U^2}{V^2} + \frac{V^2}{U^2} + 6 = \left(UV + \frac{1}{UV}\right)\left[\frac{U^3}{V^3} + \frac{V^3}{U^3} + 4\left(\frac{U}{V} + \frac{V}{U}\right)\right]. \tag{57}$$

Proof. From the page 327 of Chapter 25 of Ramanujan's second notebook [2, Entry 57, p.211], [10, Entry 57, p.327], we have

$$PQ + \frac{13}{PQ} = \frac{Q^3}{P^3} + \frac{P^3}{Q^3} - 4\left(\frac{Q}{P} + \frac{P}{Q}\right), \tag{58}$$

where $P = \frac{f(-q)}{q^{1/2}f(-q^{13})}$ and $Q = \frac{f(-q^2)}{qf(-q^{26})}$.
Replacing q by q^2 in the above equation (58), we get

$$QR + \frac{13}{QR} = \frac{R^3}{Q^3} + \frac{Q^3}{R^3} - 4\left(\frac{R}{Q} + \frac{Q}{R}\right), \tag{59}$$

where $R = \frac{f(-q^4)}{q^2f(-q^{52})}$,
which implies

$$Q^2 + \frac{13U^2}{Q^2} = aU \tag{60}$$

and

$$V^2Q^2 + \frac{13}{Q^2} = bV, \tag{61}$$

where U and V are defined as in the Theorem (2.8), $a = U^3 + \frac{1}{U^3} - \left(U + \frac{1}{U}\right)$ and $b = V^3 + \frac{1}{V^3} - \left(V + \frac{1}{V}\right)$.

Solving the above equations (60) and (61) for Q^2 and $\frac{1}{Q^2}$ and after simplification, we obtain (57). □

3. Explicit evaluation of class invariants

In this section, we prove general formula for the explicit evaluation of Weber-Ramanujan class invariants.

Define

$$g_n = 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(-q) \text{ and } G_n = 2^{-\frac{1}{4}} q^{-\frac{1}{24}} \chi(q), \tag{62}$$

where $\chi(q) = (-q; q^2)_\infty$.

Let

(63)

$$R_{k,n} = g_n g_{k^2 n}, \;\; S_{k,n} = G_n G_{k^2 n}, \;\; U_{k,n} = \frac{g_{k^2 n}}{g_n}, \;\; V_{k,n} = \frac{g_{4k^2 n}}{g_{4n}} \text{ and } W_{k,n} = \frac{G_{k^2 n}}{G_n}.$$

Theorem 3.1. *Let* $2u = W_{3,n}^6 + \frac{1}{W_{3,n}^6}$, *and* $2v = u + \sqrt{u^2 + 8}$, *then*

$$U_{3,n}^2 = \left(\sqrt{\frac{u+1}{2}} - \sqrt{\frac{u-1}{2}}\right)^{\frac{1}{3}} \left(\sqrt{\frac{v+1}{2}} + \sqrt{\frac{v-1}{2}}\right). \tag{64}$$

Proof. Using (62) and (63) in (10), we find that

$$U_{3,n}^8 W_{3,n}^4 + \frac{1}{U_{3,n}^8 W_{3,n}^4} - 6 = \left(U_{3,n}^4 W_{3,n}^2 + \frac{1}{U_{3,n}^4 W_{3,n}^2}\right) \left(W_{3,n}^6 + \frac{1}{W_{3,n}^6}\right). \tag{65}$$

Solving the above equation (65), we get

$$U_{3,n}^4 W_{3,n}^2 + \frac{1}{U_{3,n}^4 W_{3,n}^2} = 2v. \tag{66}$$

Again, solving the above equation (66) for $U_{3,n}^2$ and after simplification, we obtain (64). □

Corollary 3.2. *We have*

$$U_{3,1/3}^2 = \frac{\sqrt{3}+1}{\sqrt{2}}, \tag{67}$$

$$U_{3,1}^2 = \left(2-\sqrt{3}\right)^{\frac{1}{6}} \left(\sqrt{\frac{3+2\sqrt{3}}{2}} + \sqrt{\frac{1+2\sqrt{3}}{2}}\right), \tag{68}$$

$$U_{3,5/3}^2 = \left(\frac{3-\sqrt{5}}{2}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right). \tag{69}$$

Proof of (67). Putting $n = 1/3$ in (63), we deduce that

$$W_{3,1/3} = 1, u = 1 \text{ and } v = 2. \tag{70}$$

Using (70) in (64), we obtain (67). □

Proof of (68). Putting $n = 1$ in (63), we deduce that

$$W_{3,1} = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}, u = 2 \text{ and } v = 2 + 2\sqrt{3}. \tag{71}$$

Using (71) in (64), we obtain (68). □

Proof of (69). Putting $n = 5/3$ in (63), we deduce that

$$W_{3,5/3} = \left(\frac{3+\sqrt{5}}{\sqrt{2}}\right)^{\frac{1}{3}}, \quad u = 7/2 \text{ and } v = 4. \tag{72}$$

Using (72) in (64), we obtain (69). □

Theorem 3.3. *Let* $2s = U_{3,n}^6 - \dfrac{1}{U_{3,n}^6}$ *and* $2t = s + \sqrt{s^2-8}$, *then*

$$W_{3,n}^4 = \left(-s+\sqrt{s^2+1}\right)^{\frac{1}{3}} \left(t+\sqrt{t^2+1}\right). \tag{73}$$

Proof. Using (62) and (63) in (10), we find that

$$U_{3,n}^4 W_{3,n}^8 + \frac{1}{U_{3,n}^4 W_{3,n}^8} + 6 = \left(U_{3,n}^2 W_{3,n}^4 - \frac{1}{U_{3,n}^2 W_{3,n}^4}\right)\left(U_{3,n}^6 - \frac{1}{U_{3,n}^6}\right). \tag{74}$$

Solving the above equation (74), we find that

$$U_{3,n}^2 W_{3,n}^4 - \frac{1}{U_{3,n}^2 W_{3,n}^4} = 2t. \tag{75}$$

Again, solving the above equation (75) for $W_{3,n}^4$ and after simplification, we obtain (73). □

Corollary 3.4. *We have*

$$W_{3,2/3}^4 = \left(\sqrt{2}-1\right)^{\frac{2}{3}}\left(\sqrt{2}+\sqrt{3}\right), \tag{76}$$

$$W_{3,2}^4 = \left(\sqrt{3}-\sqrt{2}\right)^{\frac{2}{3}}\left(\sqrt{6}+\sqrt{3}+\sqrt{10+6\sqrt{2}}\right), \tag{77}$$

$$W_{3,10/3}^4 = \left(\sqrt{5}-2\right)^{\frac{2}{3}}\left(\sqrt{5}+\sqrt{6}\right)\left(2+\sqrt{3}\right), \tag{78}$$

$$W_{3,14/3}^4 = \left(2\sqrt{2}-\sqrt{7}\right)^{\frac{2}{3}}\left(\frac{\sqrt{7}+3}{\sqrt{2}}\right)\left(2+\sqrt{3}\right). \tag{79}$$

Proof of (76). Putting $n = 2/3$ in (63), we deduce that

$$W_{3,2/3} = \left(1+\sqrt{2}\right)^{\frac{1}{3}}, \quad s = 2\sqrt{2} \text{ and } t = \sqrt{2}. \tag{80}$$

Using (80) in (73), we obtain (76). □

Proof of (77). Putting $n = 2$ in (63), we deduce that

$$W_{3,2} = \left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{3}}, \quad s = 2\sqrt{6} \text{ and } t = \sqrt{6}+\sqrt{3}. \tag{81}$$

Using (81) in (73), we obtain (77). □

Proof of (78). Putting $n = 10/3$ in (63), we deduce that

$$W_{3,10/3} = \left(\sqrt{5}+2\right)^{\frac{1}{3}}, \quad s = 4\sqrt{5} \text{ and } t = 2\sqrt{5}+3\sqrt{2}. \tag{82}$$

Using (82) in (73), we obtain (78). □

Proof of (79). Putting $n = 14/3$ in (63), we deduce that

$$W_{3,14/3} = \left(2\sqrt{2}+\sqrt{7}\right), \quad s = 4\sqrt{14} \text{ and } t = 2\sqrt{14}+3\sqrt{6}. \tag{83}$$

Using (83) in (73), we obtain (79). □

Theorem 3.5. *Let* $w = \sqrt{2}\left(S_{3,n}^3 - \frac{1}{S_{3,n}^3}\right)$ *and* $2x = w + \sqrt{w^2+8}$, *then*

$$U_{3,n}^2 = \left(\sqrt{\frac{w+1}{2}}+\sqrt{\frac{w-1}{2}}\right)^{-\frac{1}{6}}\left(\sqrt{\frac{x+1}{2}}+\sqrt{\frac{x-1}{2}}\right). \tag{84}$$

Proof. From [6], we have

$$\frac{G_{9n}^6}{G_n^6}+\frac{G_n^6}{G_{9n}^6} = 2\sqrt{2}\left(G_n^3G_{9n}^3 - \frac{1}{G_n^3G_{9n}^3}\right). \tag{85}$$

Using (62) in (85), we get

$$W_{3,n}^6 + \frac{1}{W_{3,n}^6} = 2w. \tag{86}$$

Employing (86) in (65) and solving, we get

$$U_{3,n}^4 W_{3,n}^2 + \frac{1}{U_{3,n}^4 W_{3,n}^2} = 2x. \tag{87}$$

Again, solving the above equation (87), we obtain

$$U_{3,n}^2 W_{3,n} = \left(\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}}\right). \tag{88}$$

Solving the equation (86), we find that

$$W_{3,n} = \left(\sqrt{\frac{w+1}{2}} + \sqrt{\frac{w-1}{2}}\right)^{\frac{1}{6}}. \tag{89}$$

Using (89) in (88), we obtain (84). □

Corollary 3.6. *We have*

$$U_{3,1/3}^2 = \frac{\sqrt{3}+1}{\sqrt{2}}, \tag{90}$$

$$U_{3,1}^2 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{\frac{1}{6}} \left(\sqrt{\frac{2+\sqrt{3}}{2}} + \sqrt{\frac{\sqrt{3}}{2}}\right), \tag{91}$$

$$U_{3,5/3}^2 = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}} \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right). \tag{92}$$

Proof of (90). Putting $n = 1/3$ in (63), we deduce that

$$S_{3,1/3} = 2^{\frac{1}{6}}, \quad w = 1 \text{ and } x = 2. \tag{93}$$

Using (93) in (84), we obtain (90). □

Proof of (91). Putting $n = 1$ in (63), we deduce that

$$S_{3,1/3} = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{\frac{1}{3}}, \quad w = 2 \text{ and } x = 1+\sqrt{3}. \tag{94}$$

Using (94) in (84), we obtain (91). □

Proof of (92). Putting $n = 5/3$ in (63), we deduce that

$$S_{3,1/3} = \sqrt{2}, \quad w = 7/2 \text{ and } x = 4. \tag{95}$$

Using (95) in (84), we obtain (92). □

Theorem 3.7. *Let* $y = \sqrt{2}\left(R_{3,n}^3 + \frac{1}{R_{3,n}^3}\right)$ *and* $2z = y + \sqrt{y^2-8}$, *then*

$$W_{3,n}^4 = \left(y + \sqrt{y^2+1}\right)^{-\frac{1}{3}} \left(z + \sqrt{z^2+1}\right). \tag{96}$$

Proof. From [6], we have

$$\frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6} = 2\sqrt{2}\left(g_n^3 g_{9n}^3 + \frac{1}{g_n^3 g_{9n}^3}\right). \tag{97}$$

Using (62) in (97), we get

$$U_{3,n}^6 - \frac{1}{U_{3,n}^6} = 2y. \tag{98}$$

Solving the above equation, we get

$$U_{3,n} = \left(y + \sqrt{y^2+1}\right)^{\frac{1}{6}}. \tag{99}$$

Employing (98) in (74) and solving, we get

$$U_{3,n}^2 W_{3,n}^4 - \frac{1}{U_{3,n}^2 W_{3,n}^4} = 2z. \tag{100}$$

Again, solving the equation (100) and then using (98), we obtain (96). □

Corollary 3.8. *We have*

$$W_{3,2}^4 = \left(\sqrt{3} - \sqrt{2}\right)^{\frac{2}{3}} \left(\sqrt{6} + \sqrt{3} + \sqrt{10 + 6\sqrt{2}}\right), \tag{101}$$

$$W_{3,2/3}^4 = \left(\sqrt{2} - 1\right)^{\frac{2}{3}} \left(\sqrt{2} + \sqrt{3}\right), \tag{102}$$

$$W_{3,10/3}^4 = \left(9 - 4\sqrt{5}\right)^{\frac{1}{3}} \left(2 + \sqrt{3}\right)\left(\sqrt{5} + \sqrt{6}\right), \tag{103}$$

$$W_{3,14/3}^4 = \left(15 - 4\sqrt{14}\right)^{1/3} \left(\frac{\sqrt{7}+3}{\sqrt{2}}\right)\left(2 + \sqrt{3}\right) \tag{104}$$

Proof of (101). Putting $n = 2$ in (63), we deduce that

$$R_{3,2} = \left(\sqrt{3} + \sqrt{2}\right)^{\frac{1}{3}}, \quad y = 2\sqrt{6} \ \text{ and } \ z = \sqrt{6} + \sqrt{3}. \tag{105}$$

Using (105) in (96), we obtain (101). □

Proof of (102). Putting $n = 2/3$ in (63), we deduce that

$$R_{3,2} = 1, \quad y = 2\sqrt{2} \ \text{ and } \ z = \sqrt{2}. \tag{106}$$

Using (106) in (96), we obtain (102). □

Proof of (103). Putting $n = 10/3$ in (63), we deduce that

$$R_{3,10/3} = \left(\sqrt{10} + 3\right)^{\frac{1}{3}}, \quad y = 4\sqrt{5} \text{ and } z = 2\sqrt{5} + 3\sqrt{2}. \tag{107}$$

Using (107) in (96), we obtain (103). □

Theorem 3.9. *Let* $2a = W_{5,n}^3 + \frac{1}{W_{5,n}^3}$ *and* $2b = a + \sqrt{a^2 + 4}$, *then*

$$U_{5,n}^2 = W_{5,n}^{-1}\left(\sqrt{\frac{b+1}{2}} + \sqrt{\frac{b-1}{2}}\right)^2. \tag{108}$$

Proof. Using (62) and (63) in (25), we find that

$$U_{5,n}^4 W_{5,n}^2 + \frac{1}{U_{5,n}^4 W_{5,n}^2} - 2 = \left(U_{5,n}^2 W_{5,n} + \frac{1}{U_{5,n}^2 W_{5,n}}\right)\left(W_{5,n}^3 + \frac{1}{W_{5,n}^3}\right). \tag{109}$$

Solving the above equation (109), we get

$$U_{5,n}^2 W_{5,n} + \frac{1}{U_{5,n}^2 W_{5,n}} = 2b. \tag{110}$$

Again, solving the above equation, we obtain (108). □

Corollary 3.10. *We Have*

$$U_{5,1/5} = \sqrt{\frac{3+\sqrt{5}}{4}} + \sqrt{\frac{-1+\sqrt{5}}{4}}, \tag{111}$$

$$U_{5,3/5} = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right). \tag{112}$$

Proof of (111). Putting $n = 1/5$ in (63), we deduce that

$$W_{5,1/5} = 1, \quad a = 1 \text{ and } b = \frac{1+\sqrt{5}}{2}. \tag{113}$$

Using (113) in (108), we obtain (111). □

Proof of (112). Putting $n = 3/5$ in (63), we deduce that

$$W_{5,3/5} = \left(\frac{3+\sqrt{5}}{2}\right)^{\frac{1}{3}}, \quad a = 3/2 \text{ and } b = 2. \tag{114}$$

Using (114) in (108), we obtain (112). □

Theorem 3.11. *Let* $g = S_{5,n}^2 - \dfrac{1}{S_{5,n}^2}$ *and* $2h = g + \sqrt{g^2+4}$*, then*

$$U_{5,n} = \left(\sqrt{\frac{g+1}{2}} - \sqrt{\frac{g-1}{2}}\right)^{\frac{1}{3}} \left(\sqrt{\frac{h+1}{2}} + \sqrt{\frac{h-1}{2}}\right) \tag{115}$$

Proof. From [6], we have

$$2\left(G_n^2 G_{25n}^2 - \frac{1}{G_n^2 G_{25n}^2}\right) = \frac{G_{25n}^3}{G_n^3} + \frac{G_n^3}{G_{25n}^3}. \tag{116}$$

Using (62) in (116), we get

$$W_{5,n}^3 + \frac{1}{W_{5,n}^3} = 2g. \tag{117}$$

Solving the above equation (118), we get

$$W_{5,n} = \left(\sqrt{\frac{g+1}{2}} + \sqrt{\frac{g-1}{2}}\right)^{\frac{2}{3}}. \tag{118}$$

Using (117) in (109) and solving the resultant equation , we find that

$$U_{5,n}^2 W_{5,n} + \frac{1}{U_{5,n}^2 W_{5,n}} = 2h. \tag{119}$$

Again, solving the above equation (119), we obtain the required result. □

Corollary 3.12. *We have*

$$U_{5,3/5} = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{3}} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right). \tag{120}$$

Theorem 3.13. *Let* $2c = U_{5,n}^3 - \dfrac{1}{U_{5,n}^3}$ *and* $2d = c + \sqrt{c^2-4}$*, then*

$$W_{5,n}^2 = \left(-c + \sqrt{c^2+1}\right)^{\frac{1}{3}} \left(d + \sqrt{d^2+1}\right). \tag{121}$$

Proof. Using (62) and (63) in (25), we find that

$$U_{5,n}^2 W_{5,n}^4 + \frac{1}{U_{5,n}^2 W_{5,n}^4} + 2 = \left(U_{5,n} W_{5,n}^2 - \frac{1}{U_{5,n} W_{5,n}^2}\right)\left(U_{5,n}^3 - \frac{1}{U_{5,n}^3}\right). \tag{122}$$

Solving the above equation (122), we get

$$U_{5,n} W_{5,n}^2 - \frac{1}{U_{5,n} W_{5,n}^2} = 2d. \tag{123}$$

Again, solving the above equation, we obtain (121). □

Corollary 3.14. *We have*

$$W_{5,2/5}^2 = \left(\frac{-1+\sqrt{5}}{2}\right)\left(1+\sqrt{2}\right), \tag{124}$$

$$W_{5,6/5}^2 = \left(\sqrt{10}-3\right)^{\frac{1}{3}}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right). \tag{125}$$

Theorem 3.15. *Let* $e = R_{5,n}^2 + \dfrac{1}{R_{5,n}^2}$ *and* $2f = e + \sqrt{e^2+4}$, *then*

$$W_{5,n} = \left(-e+\sqrt{e^2+1}\right)^{\frac{1}{6}}\left(f+\sqrt{f^2+1}\right)^{\frac{1}{2}}. \tag{126}$$

Proof. From [6], we have

$$2\left(g_n^2 g_{25n}^2 + \frac{1}{g_n^2 g_{25n}^2}\right) = \frac{g_{25n}^3}{g_n^3} - \frac{g_n^3}{g_{25n}^3}. \tag{127}$$

Using (62) in (127), we get

$$U_{5,n}^3 - \frac{1}{U_{5,n}^3} = 2e. \tag{128}$$

Solving the above equation (128), we deduce that

$$U_{5,n} = \left(e+\sqrt{e^2+1}\right)^{\frac{1}{3}}. \tag{129}$$

Using (128) in (122) and solving the resultant equation , we find that

$$U_{5,n}W_{5,n}^2 - \frac{1}{U_{5,n}W_{5,n}^2} = 2f. \tag{130}$$

Again, solving the above equation (130), we obtain the required result (126).

□

Theorem 3.16. *Let*

$$2m = W_{7,n} + \frac{1}{W_{7,n}} \text{ and } 2n = 4m^3 + 5m + \sqrt{16m^6 + 40m^4 - 7m^2 - 42},$$

then

$$U_{7,n} = \left(\sqrt{\frac{m+1}{2}} - \sqrt{\frac{m-1}{2}}\right)\left(\sqrt{\frac{n+1}{2}} + \sqrt{\frac{n-1}{2}}\right). \tag{131}$$

Theorem 3.17. *Let* $2m = \sqrt{W_{9,n}^3} + \dfrac{1}{\sqrt{W_{9,n}^3}}$ *and* $2n = m + \sqrt{m^2+2}$, *then*

$$U_{9,n} = \left(\sqrt{\frac{m+1}{2}} - \sqrt{\frac{m-1}{2}}\right)^{\frac{1}{6}}\left(\sqrt{\frac{n+1}{2}} + \sqrt{\frac{n-1}{2}}\right). \tag{132}$$

Theorem 3.18. *Let*

$$2k = W_{13,n} + \frac{1}{W_{13,n}} \quad and \quad 2l = 4k^3 + k + \sqrt{16k^6 + 8k^4 - 3k^2 - 4},$$

then

$$U_{13,\,n} = \left(\sqrt{\frac{k+1}{2}} - \sqrt{\frac{k-1}{2}}\right)\left(\sqrt{\frac{l+1}{2}} + \sqrt{\frac{l-1}{2}}\right). \tag{133}$$

Proofs of the Theorems (3.16)-(3.18) are similar to the proof of the Theorem (3.13). So, we omit the details.

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