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Fixed-Point Theorem for Nonlinear Contractions in Digital Metric Spaces

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ABSTRACT

This study presents a fixed-point theorem in digital metric spaces under a hybrid contraction condition that combines standard distance terms with a nonlinear component involving square-root and minimum functions of distances. The proposed result generalizes classical contraction theorems, including those of Banach and Kannan offering a more flexible framework for fixed-point theory in discrete settings.

KEYWORDS: fixed point; digital image; digital metric space; contraction condition. 2020 Mathematical Sciences Classification: Primary: 54H25; Secondary: 47H10.

1. INTRODUCTION

Fixed-point theory has long been a cornerstone of mathematical analysis, providing foundational results that underpin a variety of applications in pure and applied mathematics. The Banach Contraction Principle (BCP), introduced by Stefan Banach in 1922 [1], is one of the most celebrated results in this area, offering guarantees for the presence of fixed points under specific contraction conditions. Extensions of the BCP, such as those proposed by Kannan (1968) [6], Chatterjea (1972) [3], and Reich (1971) [8], have further enriched this field by introducing diverse contraction conditions that accommodate a broader class of mappings. These classical results have proven invaluable in solving equations, optimization problems, and analyzing dynamic systems.

The introduction of digital metric spaces (DMS) by Rosenfeld (1979) [9] and its formalization by Boxer (1990) [2] marked a significant advancement in discrete topology. Unlike classical metric spaces, DMS are designed to analyze discrete structures such as digital images and lattice-based systems, bridging the gap between continuous mathematical frameworks and computational applications. In 2015, Özgür Ege and İsmet Karaca [4] extended the BCP to DMS, laying the groundwork for adapting fixed-point theory to this discrete setting. Following this, researchers like Sang Eon Han [5] and Choonkil Park et al. [7] made notable contributions by proving digital versions of Kannan, Chatterjea, and Reich contraction theorems, further advancing the field.

This study establishes a fixed-point theorem by employing a hybrid contraction condition that combines standard distance terms with a nonlinear component involving square-root and minimum functions of distances within digital metric spaces, which generalizes and extends several existing results, including the digital versions of Banach, Kannan, Chatterjea contraction theorems. This result not only unifies previous findings but also broadens their applicability to a wider class of mappings in the discrete framework.

2. PRELIMINARIES

To establish a foundation for the result outlined in this research, we detail the core definitions and attributes of digital metric spaces (DMS).

Definition 2.1 [4] "A digital metric space is a triple (X, d_k, κ) where X is a non-empty set, d_k is a metric on X satisfying non-negativity, symmetry, and the triangle inequality, κ is an adjacency relation defined on X."

Definition 2.2 [4] "The adjacency relation κ determines whether two points in X are considered neighbors. Common adjacency types include:

1. 4-adjacency in Z^2: Two points are adjacent if they share an edge.
2. 8-adjacency in Z^2: Two points are adjacent if they share either an edge or a vertex.

3. 2^n-adjacency in Z^n: Adjacency extends to n-dimensional spaces."

Definition 2.3 [4] "Let $a, b \in Z$, $a < b$. A digital interval is a set of the form $[a, b]_z = \{z \in Z \mid a \le z \le b\}$."

Definition 2.4 [4] "A k – neighbors of $p \in Z^n$ is a point of Z_n that is k - adjacent to p where $k \in \{2, 4, 8, 6, 18, 26\}$ and $n \in 1, 2, 3$. The set $N(p) = \{q \mid q$ is k - adjacent to $p\}$ is called the k -neighbourhood of p."

Definition 2.5 [1] "A map $f: (X, \kappa_0) \rightarrow (Y, \kappa_1)$ is digitally continuous if it maps every κ_0 connected subset of X into a κ_1-connected subset of Y."

Definition 2.6 [1] "A digital image (X, κ) has the fixed-point property if, for every digitally continuous map $f: X \rightarrow X$, there exists $a \in X$ such that $f(a) = a$."

Proposition 2.7 [1] "Let $\{a_n\}$ is a sequence in digital metric space (X, d_k, κ), then this sequence is said to be Cauchy sequence if and only if $\exists\ \rho \in N$ s. t.,

$$d_k(a_n, a_m) < 1, \forall\ n, m > \rho."$$

Proposition 2.8 [1] "A sequence $\{a_n\}$ in digital metric space (X, d_k, κ), converge to limit $\ell \in X$ if there is $\rho \in N$ such that $a_n = \ell$ $\forall$ n $> \rho$."

Theorem 2.9 [7] "A digital metric space (X, d_k, κ), is complete."

3. Main Result

Theorem 3.1 Let (X, d_k) be a complete digital metric space (DMS) , and $T: X \rightarrow X$ a mapping. Suppose there exist constants $k_1, k_2, k_3 \ge 0$ such that $k_1^2 + k_2^2 + k_3^2 < 1$, and for all $a, b \in X$:

$$d_k\big(T(a), T(b)\big) \le k_1^2 d_k(a, b) + k_2^2 \Big(d_k\big(a, T(a)\big) + d_k\big(b, T(b)\big)\Big) + k_3^2 \sqrt{d_k(a, b) \cdot \min\Big(d_k\big(a, T(a)\big), d_k\big(b, T(b)\big)\Big)}.$$

Then T has a unique fixed point in X.

Proof: Start with any arbitrary $a_0 \in X$. Define the sequence $\{a_n\}$ by iteratively applying T:

$$a_{n+1} = T(a_n), \quad n = 0,1,2, \ldots .$$

For any $n \ge 0$, the contraction condition gives:

$$d_k(a_{n+1}, a_n) = d_k\big(T(a_n), T(a_{n-1})\big) \le k_1^2 d_k(a_n, a_{n-1}) + k_2^2\big(d_k(a_n, a_{n+1}) + d_k(a_{n-1}, a_n)\big) + k_3^2 \sqrt{d_k(a_n, a_{n-1}) \cdot \min\big(d_k(a_n, a_{n+1}), d_k(a_{n-1}, a_n)\big)}.$$

$$d_k(a_{n+1}, a_n)(1 - k_2^2) \le k_1^2 d_k(a_n, a_{n-1}) + k_2^2 d_k(a_{n-1}, a_n) + k_3^2 \sqrt{d(a_n, a_{n-1}) \cdot \min\big(d_k(a_n, a_{n+1}), d_k(a_{n-1}, a_n)\big)}.$$

$$d_k(a_{n+1}, a_n) \le \frac{k_1^2}{1-k_2^2} d_k(a_n, a_{n-1}) + \frac{k_2^2}{1-k_2^2} d_k(a_{n-1}, a_n) + \frac{k_3^2}{1-k_2^2} \sqrt{d_k(a_n, a_{n-1}) \cdot \min\big(d_k(a_n, a_{n+1}), d_k(a_{n-1}, a_n)\big)}.$$

Since, $\sqrt{d_k(a_n, a_{n-1}) \cdot \min\big(d_k(a_n, a_{n+1}), d_k(a_{n-1}, a_n)\big)} \le d_k(a_n, a_{n-1}),$

Because, $\min\big(d_k(a_n, a_{n+1}), d_k(a_{n-1}, a_n)\big) \le d_k(a_n, a_{n-1}).$

Substituting this bound, we get:

$$d_k(a_{n+1}, a_n) \le \left(\frac{k_1^2 + k_3^2}{1 - k_2^2}\right) d_k(a_n, a_{n-1}) + \frac{k_2^2}{1 - k_2^2} d_k(a_{n-1}, a_n).$$

Define:

$$\lambda = \frac{k_1^2 + k_2^2 + k_3^2}{1 - k_2^2}.$$

Since $k_1^2 + k_2^2 + k_3^2 < 1$, it follows that $\lambda < 1$.

We now have:

$$d_k(a_{n+1}, a_n) \le \lambda d_k(a_n, a_{n-1}).$$

By induction, for any $n \ge 1$, we get:

$$d_k(a_{n+1}, a_n) \le \lambda^n d_k(a_1, a_0).$$

For any $m > n$, by using triangle inequality:

$$d_k(a_m, a_n) \le \sum_{i=n}^{m-1} d_k\,(a_{i+1}, a_i).$$

Using the iterative bound:

$$d_k(a_{i+1}, a_i) \le \lambda^i d_k(a_1, a_0),$$

we get:

$$d_k(a_m, a_n) \le \sum_{i=n}^{m-1} \lambda^i\, d_k(a_1, a_0).$$

The right-hand side is a finite geometric series:

$$\sum_{i=n}^{m-1} \lambda^i = \lambda^n \frac{1-\lambda^{m-n}}{1-\lambda}.$$

As $m, n \to \infty$, $\lambda^n \to 0$, so $d_k(a_m, a_n) \to 0$. Thus, $\{a_n\}$ is a Cauchy sequence.
Since (X, d_k) is a complete DMS, therefore, $\{a_n\}$ converges to some limit point $a_1 \in X$. That is:

$$\lim_{n\to\infty} a_n = a_1.$$

Taking the limit in $a_{n+1} = T(a_n)$, we get:

$$a_1 = T(a_1),$$

so a_1 is a fixed point of T.
Now assume, for the sake of contradiction, that a_1 and a_2 are distinct fixed points of the mapping T. Then we obtain:

$$d_k\big(T(a_1), T(a_2)\big) \le d_k(a_1, a_2) + k_2^2\left(d_k\big(a_1, T(a_1)\big) + d_k\big(a_2, T(a_2)\big)\right) + k_3^2\sqrt{d_k(a_1, a_2)\cdot \min\left(d_k\big(a_1, T(a_1)\big), d_k\big(a_2, T(a_2)\big)\right)}.$$

Since $T(a_1) = a_1$ and $T(a_2) = a_2$, substitute these into the inequality:

$$d_k\big(T(a_1), T(a_2)\big) = d_k(a_1, a_2),$$

$$d_k(a_1, a_2) \le k_1^2 d_k(a_1, a_2) + k_2^2\big(d_k(a_1, a_1) + d_k(a_2, a_2)\big) + k_3^2\sqrt{d_k(a_1, a_2)\cdot \min\big(d_k(a_1, a_1), d_k(a_2, a_2)\big)}.$$

This simplifies to:

$$d_k(a_1, a_2) \le k_1^2 d_k(a_1, a_2).$$

If $k_1^2 < 1$, dividing by $1 - k_1^2$ gives $d_k(a_1, a_2) = 0$. Hence, $a_1 = a_2$, proving uniqueness and the mapping T has a unique fixed point $a_1 \in X$.

4. CONCLUSION

The presented theorem demonstrate the strength and versatility of generalized contraction conditions within the framework of digital metric spaces (DMS). By accommodating mappings that deviate from classical conditions, this approach significantly broadens the fixed-point results. Such advancements are particularly valuable in fields like digital image processing and computer graphics, where mappings often exhibit behaviors beyond the scope of traditional contraction constraints.

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