



**STABILITY OF THE QUADRATIC FUNCTIONAL EQUATIONS  
IN THE CONTEXT OF 2-BANACH SPACES**

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ABSTRACT. The stability problem for functional equations have been extensively investigated by a number of mathematicians. During the last five decades, a number of research papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability for several functional equations, and there are interesting results related to this problem. In this research article, we investigate the generalized Hyers-Ulam stability of the functional equation  $f(2x+y)+f(x+2y) = 4f(x+y)+f(x)+f(y)$  in 2-Banach space, where  $f : X \rightarrow X$ .

**1. Introduction**

Stability of function for a function from normed space to Banach space has been studied by Hyers [9]. He has proved that for a function  $f : X \rightarrow Y$ , a function between normed space  $X$  and Banach space  $Y$  satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for each  $x, y \in X$  and  $\delta > 0$ . Then there exists a unique additive function  $T : X \rightarrow Y$  such that

$$\|f(x) - T(x)\| \leq \delta$$

for each  $x \in X$ . It is a positive answer to a problem raised by Ulam [16] for a functional equation on metric group. In fact several authors have studied the problem for different types of functional equations for functions from normed space to Banach space. (see [1, 2, 10, 11, 12, 13, 14, 15]). Stability of a functional equation for a function from a normed space to a 2-Banach space have been studied by B.M. Patel and A.B. Patel in [3], [4], [5].

Our aim is to study the stability of the functional equation

$$(1.1) \quad f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y)$$

introduced by [7], for a function on normed space to 2-Banach space.

**Definition 1.1.** Let  $X$  and  $Y$  be real vector spaces, and let  $f : X \rightarrow Y$  be a function. Then the functional equation

$$(1.2) \quad f(x + y) + f(x - y) = f(x) + f(y)$$

is said to be quadratic functional equation and every solution of the quadratic equation is said to be a quadratic function.

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**Theorem 1.2.** [7] *Let  $X$  and  $Y$  be real vector spaces. A function  $f : X \rightarrow Y$  satisfies the functional equation (1.1) if and only if function  $f : X \rightarrow Y$  satisfies the functional equation (1.2). Therefore, every solution of functional equation (1.1) is also a quadratic function.*

In the 1960s, S. Gähler introduced the concept of linear 2-normed spaces.

**Definition 1.3.** *Let  $X$  be a linear space over  $\mathbb{R}$  with  $\dim X > 1$  and let  $\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$  be a function satisfying the following properties:*

- (1)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent,
- (2)  $\|x, y\| = \|y, x\|$ ,
- (3)  $\|ax, y\| = |a|\|x, y\|$ ,
- (4)  $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for each  $x, y, z \in X$  and  $a \in \mathbb{R}$ . Then the function  $\|\cdot, \cdot\|$  is called a 2-norm on  $X$  and  $(X, \|\cdot, \cdot\|)$  is called a 2-normed space.

We introduce a basic property of 2-normed spaces as follows. Let  $(X, \|\cdot, \cdot\|)$  be a linear 2-normed space,  $x \in X$  and  $\|x, y\| = 0$  for each  $y \in X$ . Suppose  $x \neq 0$ , since  $\dim X > 1$ , choose  $y \in X$  such that  $\{x, y\}$  is linearly independent so we have  $\|x, y\| \neq 0$ , which is a contradiction. Therefore, we have the following lemma.

**Lemma 1.4.** *Let  $(X, \|\cdot, \cdot\|)$  be a 2-normed space. If  $x \in X$  and  $\|x, y\| = 0$ , for each  $y \in X$ , then  $x = 0$ .*

**Remark 1.5.** *Let  $(X, \|\cdot, \cdot\|)$  be a 2-normed space. Note that the conditions (2) and (4) imply that*

$$\|x + y, z\| \leq \|x, z\| + \|y, z\|$$

for each  $x, y, z \in X$ . Putting  $w = x + y$ , we get  $\|w, z\| \leq \|x, z\| + \|w - x, z\|$ , for each  $x, y, z \in X$ . So  $\|w, z\| - \|x, z\| \leq \|w - x, z\|$ , for each  $x, z, w \in X$ . Replacing  $w$  by  $x$  and  $x$  by  $w$  in the above inequality, we get  $\|x, z\| - \|w, z\| \leq \|x - w, z\|$  for each  $x, z, w \in X$ . Thus, we have

$$(1.3) \quad \left| \|x, z\| - \|y, z\| \right| \leq \|x - y, z\|$$

for each  $x, y, z \in X$ . Hence the function  $x \rightarrow \|x, y\|$  is continuous from  $X$  into  $\mathbb{R}$  for each fixed  $y \in X$ .

Let  $(X, \|\cdot, \cdot\|)$  be a 2-normed space. For  $x, z \in X$ , let  $p_z(x) = \|x, z\|$ ,  $x \in X$ . Then for each  $z \in X$ ,  $p_z$  is a real-valued function on  $X$  such that  $p_z(x) = \|x, z\| \geq 0$ ,  $p_z(\alpha x) = |\alpha|\|x, z\| = |\alpha|p_z(x)$  and  $p_z(x + y) = \|x + y, z\| = \|z, x + y\| \leq \|z, x\| + \|z, y\| = \|x, z\| + \|y, z\| = p_z(x) + p_z(y)$ , for each  $\alpha \in \mathbb{R}$  and all  $x, y \in X$ . Thus  $p_z$  is a semi-norm for each  $z \in X$ .

For  $x \in X$ , let  $\|x, z\| = 0$ , for each  $z \in X$ . By Lemma 1.4,  $x = 0$ . Thus for  $0 \neq x \in X$ , there is  $z \in X$  such that  $p_z(x) = \|x, z\| \neq 0$ . Hence the family  $\{p_z(x) : z \in X\}$  is a separating family of semi-norms.

Let  $x_0 \in X$ , for  $\varepsilon > 0$ ,  $z \in X$ , let  $U_{z, \varepsilon}(x_0) := \{x \in X : p_z(x - x_0) < \varepsilon\} = \{x \in X : \|x - x_0, z\| < \varepsilon\}$ . Let  $S(x_0) := \{U_{z, \varepsilon}(x_0) : \varepsilon > 0, z \in X\}$  and  $\beta(x_0) := \{\cap \mathcal{F} : \mathcal{F} \text{ is a finite subcollection of } S(x_0)\}$ . Define a topology  $\tau$  on  $X$  by saying that a set  $U$  is open if for every  $x \in U$ , there is some  $N \in \beta(x)$  such that  $N \subset U$ . That is,  $\tau$  is the topology on  $X$  that has subbase  $\{U_{z, \varepsilon}(x_0) : \varepsilon > 0, x_0 \in X, z \in X\}$ . The topology  $\tau$  on  $X$  makes  $X$  a topological

vector space. Since for  $x \in X$  collection  $\beta(x)$  is a local base whose members are convex,  $X$  is locally convex.

In the 1960s, S. Gähler and A. White introduced the concept of 2-Banach spaces.

**Definition 1.6.** A sequence  $\{x_n\}$  in a 2-normed space  $X$  is called a 2-Cauchy sequence if

$$\lim_{m,n \rightarrow \infty} \|x_n - x_m, x\| = 0$$

for each  $x \in X$

**Definition 1.7.** A sequence  $\{x_n\}$  in a 2-normed space  $X$  is called a 2-convergent sequence if there is an  $x \in X$  such that

$$\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$$

for each  $y \in X$ . If  $\{x_n\}$  converges to  $x$ , we write  $\lim_{n \rightarrow \infty} x_n = x$ .

**Definition 1.8.** We say that a 2-normed space  $(X, \|\cdot, \cdot\|)$  is a 2-Banach space if every 2-Cauchy sequence in  $X$  is 2-convergent in  $X$ .

Following shows that  $\|\cdot, \cdot\|$  is continuous in each component.

**Lemma 1.9.** For a convergent sequence  $\{x_n\}$  in a 2-normed space  $X$ ,

$$\lim_{n \rightarrow \infty} \|x_n, y\| = \|\lim_{n \rightarrow \infty} x_n, y\|$$

for each  $y \in X$ .

*Proof.* Since  $\{x_n\}$  is a 2-convergent sequence in the 2-normed space  $X$ , there is an  $x \in X$  such that  $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$  for each  $y \in X$ . By (1.3), we have

$$\lim_{n \rightarrow \infty} \left| \|x_n, y\| - \|x, y\| \right| \leq \lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$$

for each  $y \in X$ . Hence

$$\lim_{n \rightarrow \infty} \|x_n, y\| = \|x, y\| = \|\lim_{n \rightarrow \infty} x_n, y\|$$

for each  $y \in X$ . □

## 2. Stability of a functional equation for functions $f : (X, \|\cdot, \cdot\|) \rightarrow (X, \|\cdot, \cdot\|)$

Throughout this section, consider  $X$  a real normed linear space. We also consider that there is a 2-norm on  $X$  which makes  $(X, \|\cdot, \cdot\|)$  a 2-Banach space. For a function  $f : (X, \|\cdot, \cdot\|) \rightarrow (X, \|\cdot, \cdot\|)$ , define  $D_f : X \times X \rightarrow X$  by

$$D_f(x, y) = f(2x + y) + f(x + 2y) - 4f(x + y) - f(x) - f(y)$$

for each  $x, y \in X$ .

**Theorem 2.1.** Let  $\varepsilon \geq 0, 0 < p < 2, r > 0$ . Assume that the function  $f : X \rightarrow X$  satisfies

$$(2.1) \quad \|D_f(x, y), z\| \leq \varepsilon [\|x\|^p + \|y\|^p] \|z\|^r$$

for each  $x, y, z \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow X$  satisfying (1.1) and

$$(2.2) \quad \|f(x) - Q(x), z\| \leq \frac{3\varepsilon}{9 - 3^p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ .

*Proof.* Letting  $x = y = 0$  in (2.1), we have  $\|4f(0), z\| = 0$  for each  $z \in X$ , so we have  $f(0) = 0$ .

Putting  $x = y$  in (2.1), we get

$$\|2f(3x) - 4f(2x) - 2f(x), z\| \leq 2\varepsilon\|x\|^p\|z\|^r$$

for each  $x, z \in X$ . Therefore

$$(2.3) \quad \|f(3x) - 2f(2x) - f(x), z\| \leq \varepsilon\|x\|^p\|z\|^r$$

for each  $x, z \in X$ . Putting  $y = 0$  in (2.1), we get

$$(2.4) \quad \|f(2x) - 4f(x), z\| \leq \varepsilon\|x\|^p\|z\|^r$$

for each  $x, z \in X$ . Multiplying (2.4) by 2, we get

$$(2.5) \quad \|2f(2x) - 8f(x), z\| \leq 2\varepsilon\|x\|^p\|z\|^r$$

for each  $x, z \in X$ . By adding (2.3) and (2.5), we get

$$(2.6) \quad \|f(3x) - 9f(x), z\| \leq 3\varepsilon\|x\|^p\|z\|^r$$

for each  $x, z \in X$ . Therefore

$$(2.7) \quad \left\| \frac{f(3x)}{9} - f(x), z \right\| \leq \frac{\varepsilon\|x\|^p\|z\|^r}{3}$$

for each  $x, z \in X$ . Replacing  $x$  by  $3x$  in (2.7), we get

$$(2.8) \quad \left\| \frac{f(9x)}{9} - f(3x), z \right\| \leq \frac{\varepsilon 3^p\|x\|^p\|z\|^r}{3}$$

for each  $x, z \in X$ . By (2.7) and (2.8), we get

$$\begin{aligned} \left\| \frac{f(9x)}{9^2} - f(x), z \right\| &\leq \left\| \frac{f(9x)}{9^2} - \frac{f(3x)}{9}, z \right\| + \left\| \frac{f(3x)}{9} - f(x), z \right\| \\ &\leq \frac{1}{9} \frac{\varepsilon 3^p\|x\|^p\|z\|^r}{3} + \frac{\varepsilon\|x\|^p\|z\|^r}{3} \\ &= \frac{\varepsilon}{3} \left[ 1 + \frac{3^p}{9} \right] \|x\|^p\|z\|^r \end{aligned}$$

for each  $x, z \in X$ . By using induction on  $n$ , we have

$$\begin{aligned} \left\| f(x) - \frac{1}{9^n} f(3^n x), z \right\| &\leq \frac{\varepsilon\|x\|^p\|z\|^r}{3} \sum_{j=0}^{n-1} \frac{3^{pj}}{9^j} \\ &= \frac{\varepsilon\|x\|^p\|z\|^r}{3} \sum_{j=0}^{n-1} 3^{(p-2)j} \\ (2.9) \quad &= \frac{\varepsilon\|x\|^p\|z\|^r}{3} \left[ \frac{1 - 3^{(p-2)n}}{1 - 3^{p-2}} \right] \end{aligned}$$

for each  $x, z \in X$ . For  $m, n \in \mathbb{N}$ , we get

$$\begin{aligned} \left\| \frac{1}{9^m} f(3^m x) - \frac{1}{9^n} f(3^n x), z \right\| &= \left\| \frac{1}{9^{m+n-n}} f(3^{m+n-n} x) - \frac{1}{9^n} f(3^n x), z \right\| \\ &= \frac{1}{9^n} \left\| \frac{1}{9^{m-n}} f(3^{m-n} \cdot 3^n x) - f(3^n x), z \right\| \\ &\leq \frac{\varepsilon \|3^n x\|^p \|z\|^r}{3 \cdot 9^n} \sum_{j=0}^{m-n-1} 3^{(p-2)j} \\ &= \frac{\varepsilon \|x\|^p \|z\|^r}{3} 3^{(p-2)n} \sum_{j=0}^{m-n-1} 3^{(p-2)j} \\ &= \frac{\varepsilon \|x\|^p \|z\|^r}{3} \sum_{j=0}^{m-n-1} 3^{(p-2)(n+j)} \\ &= \frac{\varepsilon \|x\|^p \|z\|^r}{3} \frac{3^{(p-2)n} (1 - 3^{(p-2)(m-n)})}{1 - 3^{p-2}} \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty \end{aligned}$$

for each  $x, z \in X$ . Therefore,  $\{\frac{1}{9^n} f(3^n x)\}$  is a 2-Cauchy sequence in  $X$ , for each  $x \in X$ . Since  $X$  is a 2-Banach space,  $\{\frac{1}{9^n} f(3^n x)\}$  2-converges, for each  $x \in X$ . Define the function  $Q : X \rightarrow X$  as

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{9^n} f(3^n x)$$

for each  $x \in X$ . Now, from (2.9), we have

$$\lim_{n \rightarrow \infty} \left\| f(x) - \frac{1}{9^n} f(3^n x), z \right\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{3} \left[ \frac{1}{1 - 3^{p-2}} \right]$$

for each  $x, z \in X$ . Therefore

$$\|f(x) - Q(x), z\| \leq \frac{3\varepsilon \|x\|^p \|z\|^r}{9 - 3^p}$$

for each  $x, z \in X$ . Next we show that  $Q$  satisfies (1.1).

$$\begin{aligned} \|D_Q(x, y), z\| &= \lim_{n \rightarrow \infty} \frac{1}{9^n} \|D_f(3^n x, 3^n y), z\| \\ &= \lim_{n \rightarrow \infty} \frac{\varepsilon}{9^n} (\|3^n x\|^p + \|3^n y\|^p) \|z\|^r \\ &= \lim_{n \rightarrow \infty} \varepsilon [3^{(p-2)n} \|x\|^p + 3^{(p-2)n} \|y\|^q] \|z\|^r \\ &= 0 \end{aligned}$$

for each  $x, z \in X$ . Therefore  $\|D_Q(x, y), z\| = 0$ , for each  $z \in X$ . So we get  $D_Q(x, y) = 0$ . Next we prove the uniqueness of  $Q$ . Let  $Q'$  be another quadratic function satisfying (1.1) and (2.1). Since  $Q$  and  $Q'$  are quadratic,  $Q(3^n x) = 9^n Q(x)$ ,

$Q'(3^n x) = 9^n Q'(x)$ , for each  $x \in X$ .

$$\begin{aligned} \|Q(x) - Q'(x), z\| &= \frac{1}{9^n} \|Q(3^n x) - Q'(3^n x), z\| \\ &\leq \frac{1}{9^n} [\|Q(3^n x) - f(3^n x), z\| + \|f(3^n x) - Q'(3^n x), z\|] \\ &\leq \frac{1}{9^n} \frac{6\varepsilon \|3^n x\|^p \|z\|^r}{9 - 3^p} \\ &= 3^{(p-2)n} \frac{3\varepsilon \|x\|^p \|z\|^r}{9 - 3^p} \\ &\longrightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for each  $x, z \in X$ . Therefore  $\|Q(x) - Q'(x), z\| = 0$ , for each  $z \in X$ . Therefore  $Q(x) = Q'(x)$ , for each  $x \in X$ . This proves the uniqueness of  $Q$ .  $\square$

**Theorem 2.2.** Let  $\varepsilon \geq 0, p > 2, r > 0$ . Assume that the function  $f : X \rightarrow X$  satisfies

$$(2.10) \quad \|D_f(x, y), z\| \leq \varepsilon [\|x\|^p + \|y\|^p] \|z\|^r$$

for each  $x, y, z \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow X$  satisfying (1.1) and

$$(2.11) \quad \|f(x) - Q(x), z\| \leq \frac{3\varepsilon}{3^p - 9} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ .

*Proof.* By (2.6) of Theorem 2.1, we get

$$(2.12) \quad \|f(3x) - 9f(x), z\| \leq 3\varepsilon \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Replacing  $x$  by  $\frac{x}{3}$  in (2.12), we get

$$(2.13) \quad \left\| f(x) - 9f\left(\frac{x}{3}\right), z \right\| \leq 3\varepsilon 3^{-p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Replacing  $x$  by  $\frac{x}{3}$  in (2.13), we get

$$(2.14) \quad \left\| f\left(\frac{x}{3}\right) - 9f\left(\frac{x}{9}\right), z \right\| \leq 3\varepsilon 3^{-2p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Now, by (2.13) and (2.14)

$$\begin{aligned} \left\| f(x) - 9^2 f\left(\frac{x}{9}\right), z \right\| &\leq \left\| f(x) - 9f\left(\frac{x}{3}\right), z \right\| + \left\| 9f\left(\frac{x}{3}\right) - 9^2 f\left(\frac{x}{9}\right), z \right\| \\ &\leq 3\varepsilon 3^{-p} \|x\|^p \|z\|^r + 9 \cdot 3\varepsilon 3^{-p} \|x\|^p \|z\|^r \\ &= 3\varepsilon \|x\|^p \|z\|^r [3^{-p} + 9 \cdot 3^{-2p}] \end{aligned}$$

for each  $x, z \in X$ . By using induction on  $n \in \mathbb{N}$ , we get

$$\begin{aligned} \left\| f(x) - 9^n f\left(\frac{x}{3^n}\right), z \right\| &\leq 3\varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 3^{-p(j+1)} \cdot 9^j \\ &= 3\varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 3^{(-p+2)j-p} \\ (2.15) \quad &= 3\varepsilon \|x\|^p \|z\|^r \frac{3^{-p}(1 - 3^{(-p+2)n})}{1 - 3^{-p+2}} \end{aligned}$$

for each  $x, z \in X$ . Now, for  $m, n \in \mathbb{N}$ , we get

$$\begin{aligned} \left\| 9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), z \right\| &= \left\| 9^{m+n-n} f\left(\frac{x}{3^{m+n-n}}\right) - 9^n f\left(\frac{x}{3^n}\right), z \right\| \\ &= 9^n \left\| 9^{m-n} f\left(\frac{x}{3^{m-n} \cdot 3^n}\right) - f\left(\frac{x}{3^n}\right), z \right\| \\ &\leq 3\varepsilon 9^n \left\| \frac{x}{3^n} \right\|^p \|z\|^r \sum_{j=0}^{m-n-1} 3^{(-p+2)j-p} \\ &= 3\varepsilon 3^{(-p+2)n} \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 3^{(-p+2)j-p} \\ &= 3\varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 3^{(-p+2)(n+j)-p} \\ &= 3\varepsilon \|x\|^p \|z\|^r \frac{3^{(-p+2)n} (1 - 3^{(-p+2)(m-n)})}{1 - 3^{-p+2}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for each  $x, z \in X$ . Therefore  $\left\{ 9^n f\left(\frac{x}{3^n}\right) \right\}$  is a Cauchy sequence in  $X$ . Since  $X$  is a 2-Banach space,  $\left\{ 9^n f\left(\frac{x}{3^n}\right) \right\}$  converges in  $X$ . So, define  $Q : X \rightarrow X$  as

$$Q(x) := \lim_{n \rightarrow \infty} 9^n f\left(\frac{x}{3^n}\right)$$

for each  $x \in X$ . Now, by (2.15), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| f(x) - 9^n f\left(\frac{x}{3^n}\right), z \right\| &\leq 3\varepsilon \|x\|^p \|z\|^r \frac{3^{-p}}{1 - 3^{-p+2}} \\ &= 3\varepsilon \|x\|^p \|z\|^r \frac{1}{3^p - 9} \end{aligned}$$

for each  $x, z \in X$ . Therefore

$$\|f(x) - Q(x), z\| \leq 3\varepsilon \|x\|^p \|z\|^r \frac{1}{3^p - 9}$$

for each  $x, z \in X$ . The rest of the proof is similar to the proof of Theorem 2.1.  $\square$

**Theorem 2.3.** Let  $\varepsilon \geq 0, 0 < p < 2$ . Assume that the function  $f : X \rightarrow X$  satisfies

$$(2.16) \quad \|D_f(x, y), z\| \leq \varepsilon [\|x\|^p + \|y\|^p] \|z\|^r$$

for each  $x, y, z \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow X$  satisfying (1.1) and

$$(2.17) \quad \|f(x) - Q(x), z\| \leq \frac{\varepsilon}{4 - 2^p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ .

*Proof.* By (2.4) of Theorem 2.1, we get

$$\|f(2x) - 4f(x), z\| \leq \varepsilon \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Therefore

$$(2.18) \quad \left\| \frac{f(2x)}{4} - f(x), z \right\| \leq \frac{\varepsilon}{4} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Replacing  $x$  by  $2x$  in (2.18), we get

$$(2.19) \quad \left\| \frac{f(4x)}{4} - f(2x), z \right\| \leq \frac{\varepsilon}{4} 2^p \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . By (2.18) and (2.19), we get

$$\begin{aligned} \left\| \frac{f(4x)}{4^2} - f(x), z \right\| &\leq \left\| \frac{f(4x)}{4^2} - \frac{f(2x)}{4}, z \right\| + \left\| \frac{f(2x)}{4} - f(x), z \right\| \\ &\leq \frac{1}{4} \cdot \frac{\varepsilon}{4} 2^p \|x\|^p \|z\|^r + \frac{\varepsilon}{4} \|x\|^p \|z\|^r \\ &= \frac{\varepsilon}{4} \|x\|^p \|z\|^r \left[ 1 + \frac{2^p}{4} \right] \end{aligned}$$

for each  $x, z \in X$ . Now, by using induction on  $n \in \mathbb{N}$ , we get

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - f(x), z \right\| &\leq \frac{\varepsilon}{4} \|x\|^p \|z\|^r \sum_{j=0}^{n-1} \frac{2^{pj}}{4^j} \\ &= \frac{\varepsilon}{4} \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 2^{(p-2)j} \\ (2.20) \quad &= \frac{\varepsilon}{4} \|x\|^p \|z\|^r \left[ \frac{1 - 2^{(p-2)n}}{1 - 2^{p-2}} \right] \end{aligned}$$

for each  $x, z \in X$ . For  $m, n \in \mathbb{N}$ , by (2.20), we get

$$\begin{aligned} \left\| \frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}, z \right\| &= \left\| \frac{f(2^{m+n-n} x)}{4^{m+n-n}} - \frac{f(2^n x)}{4^n}, z \right\| \\ &= \frac{1}{4^n} \left\| \frac{f(2^{m-n} \cdot 2^n x)}{4^{m-n}} - f(2^n x), z \right\| \\ &\leq \frac{1}{4^n} \frac{\varepsilon}{4} \|2^n x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(p-2)j} \\ &= \frac{\varepsilon}{4} 2^{(p-2)n} \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(p-2)j} \\ &= \frac{\varepsilon}{4} \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(p-2)(n+j)} \\ &= \frac{\varepsilon}{4} \|x\|^p \|z\|^r \frac{2^{(p-2)n} (1 - 2^{(p-2)(m-n)})}{1 - 2^{p-2}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for each  $x, z \in X$ . Therefore  $\left\{ \frac{f(2^n x)}{4^n} \right\}$  is a Cauchy sequence in  $X$ . Since  $X$  is a 2-Banach space,  $\left\{ \frac{f(2^n x)}{4^n} \right\}$  converges in  $X$ . So, define  $Q : X \rightarrow X$  as

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for each  $x \in X$ . Now, by (2.20), we get

$$\lim_{n \rightarrow \infty} \left\| \frac{f(2^n x)}{4^n} - f(x), z \right\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{4(1 - 2^{p-2})}$$

for each  $x, z \in X$ . Therefore

$$\|Q(x) - f(x), z\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{4 - 2^p}$$

for each  $x, z \in X$ . The rest of the proof is similar to the proof of Theorem 2.1.  $\square$

**Theorem 2.4.** Let  $\varepsilon \geq 0, p > 2, r > 0$ . Assume that a function  $f : X \rightarrow X$  satisfies

$$(2.21) \quad \|D_f(x, y), z\| \leq \varepsilon [\|x\|^p + \|y\|^p] \|z\|^r$$

for each  $x, y, z \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow X$  satisfying (1.1) and

$$(2.22) \quad \|f(x) - Q(x), z\| \leq \frac{\varepsilon}{2^p - 4} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ .

*Proof.* By (2.4) of Theorem 2.1, we get

$$\|f(2x) - 4f(x), z\| \leq \varepsilon \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Therefore

$$(2.23) \quad \|f(2x) - 4f(x), z\| \leq \varepsilon \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Replacing  $x$  by  $\frac{x}{2}$  in (2.23), we get

$$(2.24) \quad \left\| f(x) - 4f\left(\frac{x}{2}\right), z \right\| \leq \varepsilon 2^{-p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . Replacing  $x$  by  $\frac{x}{2}$  in (2.24), we get

$$(2.25) \quad \left\| f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), z \right\| \leq \varepsilon 2^{-2p} \|x\|^p \|z\|^r$$

for each  $x, z \in X$ . By (2.24) and (2.25), we get

$$\begin{aligned} \left\| f(x) - 16f\left(\frac{x}{4}\right), z \right\| &\leq \left\| f(x) - 4f\left(\frac{x}{2}\right), z \right\| + \left\| 4f\left(\frac{x}{2}\right) - 16f\left(\frac{x}{4}\right), z \right\| \\ &\leq \varepsilon 2^{-p} \|x\|^p \|z\|^r + 4\varepsilon 2^{-2p} \|x\|^p \|z\|^r \\ &= \varepsilon \|x\|^p \|z\|^r [2^{-p} + 4 \cdot 2^{-2p}] \end{aligned}$$

for each  $x, z \in X$ . By using induction on  $n \in \mathbb{N}$ , we get

$$\begin{aligned} \left\| f(x) - 4^n f\left(\frac{x}{2^n}\right), z \right\| &\leq \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 2^{-p(j+1)} \cdot 4^j \\ &= \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 2^{(-p+2)j-p} \\ (2.26) \quad &= \varepsilon \|x\|^p \|z\|^r \frac{2^{-p}(1 - 2^{(-p+2)n})}{1 - 2^{-p+2}} \end{aligned}$$

for each  $x, z \in X$ . For  $m, n \in \mathbb{N}$ ,  $m > n$ , we get

$$\begin{aligned} \left\| 4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), z \right\| &= \left\| 4^{m+n-n} f\left(\frac{x}{2^{m+n-n}}\right) - 4^n f\left(\frac{x}{2^n}\right), z \right\| \\ &= 4^n \left\| 4^{m-n} f\left(\frac{x}{2^{m-n} \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right), z \right\| \\ &\leq 4^n \varepsilon \left\| \frac{x}{2^n} \right\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\ &= \varepsilon 2^{(-p+2)n} \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\ &= \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(-p+2)(n+j)-p} \\ &= \varepsilon \|x\|^p \|z\|^r \frac{2^{(-p+2)n-p} (1 - 2^{(-p+2)(m-n)})}{1 - 2^{(-p+2)}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

for each  $x, z \in X$ . Therefore  $\left\{ 4^n f\left(\frac{x}{2^n}\right) \right\}$  is a Cauchy sequence in  $X$ . Since  $X$  is a 2-Banach space,  $\left\{ 4^n f\left(\frac{x}{2^n}\right) \right\}$  converges in  $X$ . So, define  $Q : X \rightarrow X$  as

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for each  $x \in X$ . Now, by (2.26), we have

$$\left\| f(x) - 4^n f\left(\frac{x}{2^n}\right), z \right\| \leq \varepsilon \|x\|^p \|z\|^r \frac{2^{-p} (1 - 2^{(-p+2)n})}{1 - 2^{-p+2}}$$

for each  $x, z \in X$ . Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| f(x) - 4^n f\left(\frac{x}{2^n}\right), z \right\| &\leq \varepsilon \|x\|^p \|z\|^r \frac{2^{-p}}{1 - 2^{-p+2}} \\ &= \varepsilon \|x\|^p \|z\|^r \frac{1}{2^p - 4} \end{aligned}$$

for each  $x, z \in X$ . Therefore

$$\|f(x) - Q(x), z\| \leq \varepsilon \|x\|^p \|z\|^r \frac{1}{2^p - 4}$$

for each  $x, z \in X$ . The rest of the proof is similar to the proof of Theorem 2.1.  $\square$

#### REFERENCES

1. J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. 80(1980), 411-416.
2. J. H. Bae and K. W. Jun, *On the generalized Hyers-Ulam-Rassias stability of an n-dimensional quadratic functional equation*, J. Math. Anal. Appl. 258(2001), 183-193.
3. B.M. Patel and A.B. Patel, *Stability of Quartic Functional equations in 2-Banach space*, *Int. Journal of Math. Analysis*, 7(2013), No. 23, 1097-1107.
4. B.M. Patel and A.B. Patel, *Stability of Quadratic Functional equations in 2-Banach space*, *Gen. Math. Notes*, 15(2013), No. 2.
5. B.M. Patel and A.B. Patel, *Stability of Euler-Lagrange Quadratic Functional equations in 2-Banach space*, *International Journal of Pure and Applied Mathematical Sciences*, 6(2013), No. 4, 333-353.

6. Bhavin Mansukhlal Patel, *Stability of additive and a generalized quadratic functional equations in Banach spaces*, Asian Research Journal of Mathematics, Vol 21, Issue 11,36-51, Article no. ARJOM.146955
7. I. S. Chang and H. M. Kim, *On the Hyers-Ulam- stability of a quadratic functional equations*, J. of Inequalities in Pure and Applied Mathematics, 3(3)(2002), No. 33.
8. S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr., 26(1963) 115-148.
9. D. H. Hyers, *On the Stability of the Linear Functional Equation*, Proc. Nat. Acad. Sci. U.S.A. 27(1941), 222-224.
10. D. H. Hyers, G. Isac, and Th. M. Rassias, *"Stability of Functional Equations in Several variables"*, Birkhäuser, Basel, 1998.
11. D. H. Hyers, G. Isac, and Th. M. Rassias, *On the asymptoticity aspect of Hyers-Ulam stability of mappings*, Proc. Amer. Math. Soc. 126(1998), 425-430.
12. D. H. Hyers and Th. M. Rassias, *Approximate homomorphism*, Aequationes Math.44(1992),125-153.
13. K. W. Jun and H. M. Kim, *Remarks on the stability of additive functional equation*, Bull. Korean Math. soc. 38(2001), 679-687.
14. K. W. Jun and Y. H. Lee, *On the Hyers-Ulam-Rassias stability of a generalized quadratic equation*, Bull. Korean Math. soc. 38(2001), 261-272.
15. J. M. Rassias, *On Approximation of Approximately Linear mappings by Linear mappings*, Bull. Sci. Math. 108 (1984), 445-446.
16. S. M. Ulam, *Problems in Modern Mathematics*, Chap. VI, Science ed. Wiley, New York, 1964.

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