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Derivation|Correction of Hardy-Littlewood Twin Prime Constant using Prime Generator Theory (PGT)

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Abstract: The Hardy–Littlewood twin prime constant is a metric to compute the distribution of twin primes. Using Prime Generator Theory (PGT) it is shown it is more easily mathematically and conceptually derived, and the correct value is a factor of 2 larger.

Introduction

In the early 1920's the British mathematicians, Godfrey Harold (G.H.) Hardy and John Edensor Littlewood teamed up to write (among a series of their many collaborations) on the topic of the Twin Primes problems. It will be shown their ***twin prime constant*** is off by a factor of 2, and its computation is conceptually simple, and mathematical easy to derive, using *Prime Generator Theory* (*PGT)*.

First Hardy–Littlewood Conjecture

The Hardy–Littlewood conjecture is a generalization of the Twin Prime conjecture. It's concerned with the distribution of *prime constellations*, including twin primes, in analogy to the *Prime Number Theory*.

A synopsis of its statement given by the Wiki2 article on *Polignac's Conjecture* [5] provides:

Conjectured Density

Let $\pi_n(x)$. (*n* even) be the number of prime gaps of size *n* below *x*.

The first Hardy–Littlewood conjecture says the asymptotic density is of form

(1)
$$\pi_2(x) \sim 2C_2 \frac{x}{(\ln x)^2} \sim 2C_2 \int_2^x \frac{dt}{(\ln t)^2}$$

Here C_2, called the twin primes constant, is defined as follows:

(2)
$$C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.66016181584686957392781211001 4\ldots$$

Polignac's Conjecture (1849) states – there are an infinite number of prime pairs that differ by any even value *n*. The *Twin Primes Conjecture* is for the specific case of prime gaps of 2, e.g. (5, 7) and (11, 13).

In a previous paper|video [1],[2] I established using *PGT* that *Polignac's Conjecture* is true. I'll first present some of its basic concepts, and its mathematical framework, that will be used to construct and explain the reasoning for the corrected Hardy-Littlewood (*H-L*) twin prime constant.

Modular Groups

Prime Generator Theory is derived from the properties of modular groups of size $\mathbb{Z}_n$, for even values *n*. When the number line is broken into successive groups of size *n*, the integers $\boldsymbol{r}_i < \boldsymbol{n}$ that are coprime to it are its ***residues***. They are the coprime primes (and possibly their multiples) not prime factors of *n*.

The number of residues of *n* are determined by the *Euler Totient Function (ETF)*, which has *PGT* form:

(3) $$\varphi(n) = n \prod_{i=1}^{j} \frac{(p_i - 1)}{p_i}$$

where the p_i are the *j* unique prime factors of *n*. For primorial values *n* = $p_m\#$ (the product of the first *m* primes), the residues count – ***rescntpn*** – for *n* = $p_m\#$ simply becomes:

(4) $$rescntpn = \prod_{i=1}^{m}(p_i - 1) = (p_m - 1)\#$$

The expression $(p_m - 1)\#$ (also written as $P_m^{-1}\#$) is the ***first reduced primorial*** of the first *m* primes. Reduced primorials have general form $(p_m - r)\#$ or $P_m^{-r}\#$ for *r* ∈ {1...p_m}, with $(0)\#$ = 1, for *r* = p_m. The principal and reduced primorials play a central role in the structure and the foundation of *PGT.*

Thus, *n* is the modulus – ***modpn*** – of $\mathbb{Z}_n$, of the integers {0...*n*-1}, whose residues count – ***rescntpn*** – is given by $\varphi(n)$, whose ***residues*** values are the coprime integers $\boldsymbol{r}_i$ to ***modpn***, e.g. *gcd*($\boldsymbol{r}_i$, ***modpn***) = 1.

Residue Gaps

As we create successively larger primorial modular groups, which contain more prime residues, the gap patterns between the residues completely characterize the gaps between the primes. The a_i residue gap coefficients hold the residue gaps counts, e.g. a_i is the number of residue gaps of size 2*i* for *n* = $p_m\#$. The table below shows the first 11 primorial groups moduli, their residue counts $P_m^{-1}\#$, and gap counts of sizes 2|4, i.e. $a_1|a_2$, which are equal and odd, of ***second reduced primorial*** form: $a_{1|2} = (p_m - 2)\#$.

m	1	2	3	4	5	6	7	8	9	10	11
p_m	2	3	5	7	11	13	17	19	23	29	31
$p_m\#$	2	6	30	210	2,310	30,030	510,510	9,699,690	223,092,870	6,469,693,230	200,560,490,130
$(p_m - 1)\#$	1	2	8	48	480	5,760	92,160	1,658,880	36,495,360	1,021,870,080	30,656,102,400
$(p_m - 2)\#$	1	1	3	15	135	1,485	22,275	378,675	7,952,175	214,708,725	6,226,553,025

Figure 1.

From this simple deterministic mathematical framework, it is conceptually easy to see and understand, and mathematically derive, the physical modular group meaning of the twin prime constant. We will see it's the product of two ratios that define physical properties of primorial modular groups, whose correct value is a factor of 2 larger than that derived by Hardy-Littlewood.

Below is the picture for $\mathbb{Z}_{30}$, where $n = p_3\# = 5\# = 30$, and $\varphi(30) = 8$. For $\mathbb{Z}_{30}$, its canonical residues are {1, 7, 11, 13, 17, 19, 23, 29}, with prime values: $\boldsymbol{p_m = 30k + r_i}$, $k \geq 0$, i = 0..7, and each $\boldsymbol{p_m \equiv r_i \bmod 30}$.

0	30	60	90	120	150	180	210	240	270	300	330	360	390	420	450	480	510	540	570	
1	31	61	91	121	151	181	211	241	271	301	331	361	391	421	451	481	511	541	571	….
2	32	62	92	122	152	182	212	242	272	302	332	362	392	422	452	482	512	542	572	
3	33	63	93	123	153	183	213	243	273	303	333	363	393	423	453	483	513	543	573	
4	34	64	94	124	154	184	214	244	274	304	334	364	394	424	454	484	514	544	574	6
5	35	65	95	125	155	185	215	245	275	305	335	365	395	425	455	485	515	545	575	
6	36	66	96	126	156	186	216	246	276	306	336	366	396	426	456	486	516	546	576	
7	37	67	97	127	157	187	217	247	277	307	337	367	397	427	457	487	517	547	577	….
8	38	68	98	128	158	188	218	248	278	308	338	368	398	428	458	488	518	548	578	
9	39	69	99	129	159	189	219	249	279	309	339	369	399	429	459	489	519	549	579	4
10	40	70	100	130	160	190	220	250	280	310	340	370	400	430	460	490	520	550	580	
11	41	71	101	131	161	191	221	251	281	311	341	371	401	431	461	491	521	551	581	….
12	42	72	102	132	162	192	222	252	282	312	342	372	402	432	462	492	522	552	582	2
13	43	73	103	133	163	193	223	253	283	313	343	373	403	433	463	493	523	553	583	….
14	44	74	104	134	164	194	224	254	284	314	344	374	404	434	464	494	524	554	584	
15	45	75	105	135	165	195	225	255	285	315	345	375	405	435	465	495	525	555	585	**4**
16	46	76	106	136	166	196	226	256	286	316	346	376	406	436	466	496	526	556	586	
17	47	77	107	137	167	197	227	257	287	317	347	377	407	437	467	497	527	557	587	….
18	48	78	108	138	168	198	228	258	288	318	348	378	408	438	468	498	528	558	588	2
19	49	79	109	139	169	199	229	259	289	319	349	379	409	439	469	499	529	559	589	….
20	50	80	110	140	170	200	230	260	290	320	350	380	410	440	470	500	530	560	590	
21	51	81	111	141	171	201	231	261	291	321	351	381	411	441	471	501	531	561	591	4
22	52	82	112	142	172	202	232	262	292	322	352	382	412	442	472	502	532	562	592	
23	53	83	113	143	173	203	233	263	293	323	353	383	413	443	473	503	533	563	593	….
24	54	84	114	144	174	204	234	264	294	324	354	384	414	444	474	504	534	564	594	
25	55	85	115	145	175	205	235	265	295	325	355	385	415	445	475	505	535	565	595	
26	56	86	116	146	176	206	236	266	296	326	356	386	416	446	476	506	536	566	596	6
27	57	87	117	147	177	207	237	267	297	327	357	387	417	447	477	507	537	567	597	
28	58	88	118	148	178	208	238	268	298	328	358	388	418	448	478	508	538	568	598	
29	59	89	119	149	179	209	239	269	299	329	359	389	419	449	479	509	539	569	599	….

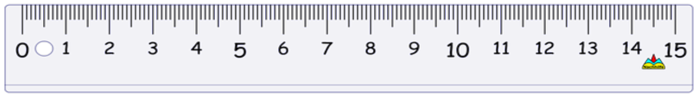
Here we see, per Figure 1., the isolated 3 twin residues pairs, and the 25 twin primes values along them. Including the only consecutive first twins (3, 5) and (5, 7), they constitute the 27 twin primes ≤ 618.

11	41	71	101	131	161	191	221	251	281	311	341	371	401	431	461	491	521	551	581	….
																				2
13	43	73	103	133	163	193	223	253	283	313	343	373	403	433	463	493	523	553	583	….
17	47	77	107	137	167	197	227	257	287	317	347	377	407	437	467	497	527	557	587	….
																				2
19	49	79	109	139	169	199	229	259	289	319	349	379	409	439	469	499	529	559	589	….
29	59	89	119	149	179	209	239	269	299	329	359	389	419	449	479	509	539	569	599	….
																				2
31	61	91	121	151	181	211	241	271	301	331	361	391	421	451	481	511	541	571	601	….

Thus, ***we explicitly see*** the twin (cousin, etc) primes are uniquely characterized, and identified, from the deterministic residues gap structure of modular groups $\mathbb{Z}_n$, particularly for n a primorial, i.e. $n = p_m\#$.

Ruler Rules

A useful visual metaphor is to imagine the modular group size – ***modpn*** – as a ruler with integer units.



Rulers consists of gaps of a_i sizes and markers to designate their size. Here the little marks are all the group's integers and the big marks residues. The gaps and markings adhere to the 2 basic ***Ruler Rules***:

RR1: total length = sum of all gaps = Σgaps

RR2: number of residues = number of gaps = Σa_i

For *n* = $p_m\#$, the average gap size (integers per gap) is the length divided by the number of residues. The rulers are $P_m\#$ integers long with $P_m^{-1}\#$ residues|gaps, thus the average integers/residue-gap are:

(5)
$$\hat{g}_n = \frac{P_m\#}{P_m^{-1}\#} = \frac{p_m\#}{(p_m-1)\#}$$

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\hat{g}_n$	2	3	3.75	4.375	4.8125	5.214	5.539	5.847	6.113	6.331	6.542	6.724	6.892	7.056

Figure 2.

Here, each larger p_m ratio is decreasing, approaching 1.0 from above, i.e. $p_m/(p_m-1) \rightarrow$ 1.00000... However, the product of their successive ratios grows (slowly) infinitely larger in value as $p_m \rightarrow \infty$.

Twins|Cousins Ratios

Twin and Cousin primes are prime pairs that differ by 2 and 4. The coefficients a_1 and a_2 indicate their pair candidates count within a modular group. Thus their primorial pairs count values – ***pairstcntpn*** – is given by $(p_m-2)\#$, which are the number of residue pair gaps of 2 and 4 for $\mathbb{Z}_n$, when *n* = $p_m\#$.

Thus, let ***r₂*** be the percentage of gap sizes of 2|4 among the total number of residue gaps, given by:

(6)
$$r_2 = \frac{2|4\ gaps\ count}{total\ gaps\ count} = \frac{(p_m-2)\#}{(p_m-1)\#}$$

Here, each larger p_m ratio is increasing towards 1.0 from below, i.e. $(p_m-2)/(p_m-1) \rightarrow$ 0.99999…., with their successive ratio products becoming smaller $\rightarrow$ 0, with each additional prime.

PGT Derivation

From (1), notice the factor 2 attached to C_2 exists because *H-L* used only the odd primes, and excluded the first prime p_1 = 2 from the primes products values. This is probably because of the (*p* – 2) reduced prime factor in the numerator of (2), whose literal arithmetic computation for *p* = 2 makes everything zero. However, *PGT* informs us that conceptually the numerator|denominator multiplications in (2) are primorials, which represent physical modular group characteristics.

When p_1 = 2 is included in (2), the numerator|denominator expressions become primorials over all *p*.

(7) $$C_2 = \frac{p_m\#(p_m - 2)\#}{(p_m - 1)\#^2}$$

And for p_1 = 2 we get the added factor of 2 in (1).

(8) $$C_2 = \frac{2\#(0)\#}{(1)\#^2} = \frac{2(1)}{1} = 2$$

Thus C_2 as written, can be conceptually understood as the ratio of these physical parameters.

(9) $$C_2 = \frac{modpn \cdot pairscntpn}{rescntpn^2}$$

But this can be broken into the product of 2 rational ratios of physical primorial group quantities.

(10) $r_1 = \frac{modpn}{rescntpn} = \frac{p_m\#}{(p_m - 1)\#}$ the (increasing) average modular residue gap size $\hat{g}_n$

(11) $r_2 = \frac{pairscntpn}{rescntpn} = \frac{(p_m - 2)\#}{(p_m - 1)\#}$ the (decreasing) percentage|% of twin|cousin pair gaps

The conceptual and physical meaning of this is now easy to understand. As the residues|gaps increase as more primes are used to form the $\mathbb{Z}_n$ modular groups, the average integer distance|gaps between the residues increase, which for gaps of 2 and 4, is offset by a proportional countervailing decrease in their residue gaps percentage of the total number of gaps, such that their product is constant.

Thus r_1 will grow (slowly) without end, while r_2 decreases to near zero without end. Their product C_2, however, approaches an increasing stable equilibrium (more unchanging digits) as more primes are used to form the $\mathbb{Z}_n$ modular groups.

PGT informs us, as we use more primes to separate the number line into groups of *n* = $p_m\#$ integers, the primes are squeezed into a decreasing percentage of the integer number space along the increasing residues. Thus the residue gaps increasingly become gaps strictly between prime residues. Because gaps of 2 and 4 are ***atomic***, and mathematically expressible as primorial values, their percentage is a constant ratio of all the residues gaps, and thus also for Twin and Cousin primes. And as there are an infinite number of primes, then so too for any even gap size value between their pairs [1],[2].

This Ruby code computes r_1, r_2, C_2 using 1,000 primes, which can be increased for more stable digits.

```
require "primes/utils"         # external rubygem by Jabari Zakiya
                               # install first with $ gem install primes-utils
primes1000 = (1000.nthprime).primes # Array of first 1,000 primes (2..7919)

# Compute ratios: r1, r2, and c2, and print values for every 100th prime
r1, r2 = 2, 1                  # initialize values for first prime p=2

# Start using array primes from 2nd prime p=3, i.e. primes1000[1]
primes1000[1..].each_with_index do |p, i|
   p_1 = (p - 1.0)             # make p_1 a floating point value
   r1 *= p / p_1               # update  r1 for current prime
   r2 *= (p - 2) / p_1         # update  r2 for current prime
   c2  = r1 * r2               # compute c2 for current prime
   pth_prime = i + 2           # prmth val for current prime
   if pth_prime % 100 == 0     # if prmth value a multiple of 100 output results
      puts "\nUpto #{pth_prime}th prime #{p}"
      puts "r1 = #{r1} \nr2 = #{r2} \nC2 = #{c2} \n"
   end
end
```

This table shows the computed data produced by the code for the given number of primes shown.

m primes	r1	r2	C2
100	11.267620389582685	0.11720847212721400	1.3206605703724303
200	12.714305399732526	0.10385598009615430	1.3204566485310485
300	13.553362181029541	0.09742244361035730	1.3204016628121005
400	14.143788321617125	0.09335387299959709	1.3203774187094295
500	14.600162046719854	0.09043488777762677	1.3203640162302757
600	14.972683892976060	0.08818429692348541	1.3203556021596883
700	15.285994047978852	0.08637645148081055	1.3203499232212041
800	15.556739346163312	0.08487291685854093	1.3203458451169110
900	15.795529761036450	0.08358964822246509	1.3203427762125148
1000	16.008556779361957	0.08247716653126569	1.3203404034166584
500M	41.186500241230260	0.03205719407975786	1.3203236316991123

Figure 3.

The last row for 500M primes (up to 11,037,271,757) is shown to provide some perspective on growth. Below are the *H-L* and *PGT twin prime constant* values, and computed *PGT* value for 500M primes.

H-L: C2 = 0.660161815846869573927812110014...
PGT: C2 = 1.320323631693739147855624220028...
500M: C2 = 1.3203236316991123

Thus we see 500M primes give 12 accurate|stable digits, which will increase as more primes are used. Using languages|code that provide higher floating point precision can provide more accurate digits.

A plot of the log values of the data highlights better the slow rates of change for r_1 and r_2 , which is why after 500M primes C_2 only has 12 accurate|stable digits.

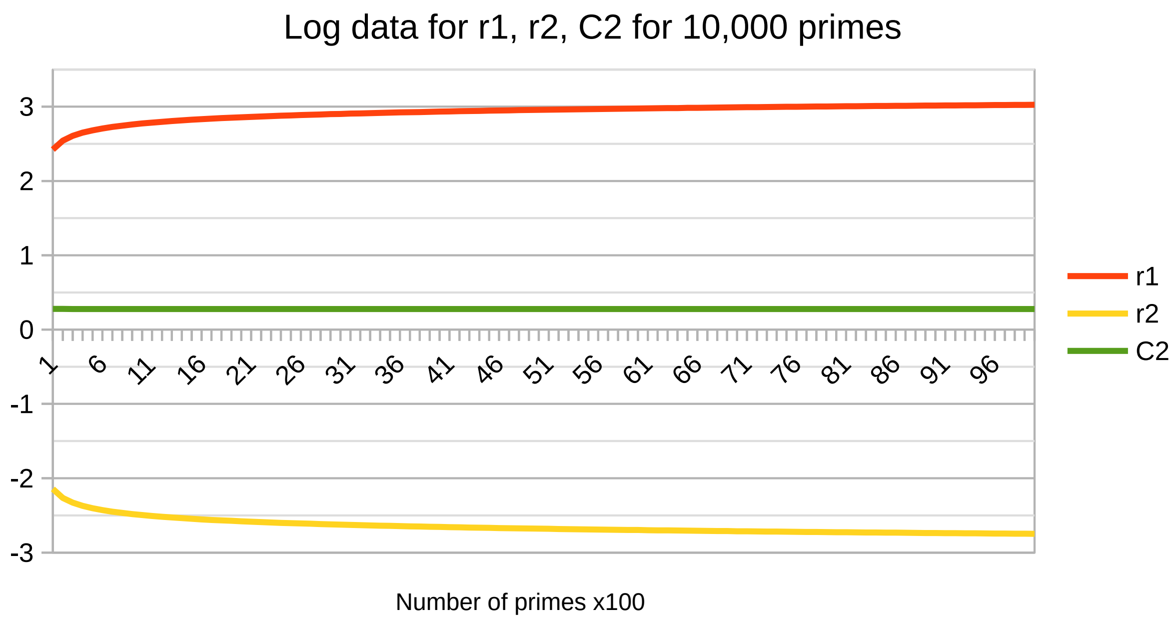


Figure 4.

Conclusion

When Hardy-Littlewood constructed their multiplicative form for the *twin prime constant* they used only the odd primes, excluding the first prime 2. This was likely because of the (*p* – 2) factor in the numerator, which they didn't know how to conceptually deal with. However, they realized they had to include a factor of 2 to the twin primes distribution integral to make the computations work.

Applying the conceptual understanding that *PGT* provides of its mathematical framework derived from modular groups, we see and understand that the external factor of 2 Hardy-Littlewood attached to their constant is actually a part of it, upon the realization that the prime multiplications can be conceptually treated, and computed, as multiplications of primorials.

We further see these primorial forms represent physical characteristics of modular groups $\mathbb{Z}_n$, when we break the number line into successive size groups of *n* = $p_m\#$ integers. Then, residue gaps of 2 and 4 completely characterize the distribution for Twin and Cousin primes (as well for the other residue gap size coefficients profiles do for the infinite prime pairs that differ by their values).

Thus, *PGT* provides a conceptually simple, yet mathematically powerful, framework to understand and characterize the distribution and relationship of primes, using only elementary concepts, arithmetic, and logic.

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