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Theorems on Common Fixed Points in Complete Rectangular S-Metric Spaces

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ABSTRACT
In this paper, some common fixed point theorems are established in rectangular S-metric spaces, an advanced generalization of S-metric spaces. New common fixed point theorems are developed that integrate and extend various well known results in fixed point theory. The findings are further supported by illustrative examples.

KEYWORDS: Rectangular metric spaces, S-metric spaces, Rectangular S-metric spaces.

INTRODUCTION

One of the most important topics in the development of non linear analysis is fixed point theory. Fixed point theory has also been successfully applied to a variety of other fields of research, including chemistry, biology, economics, computer science, engineering and a variety of others. The concept of metric spaces has been generalized in various ways. The ideas of 2-metric spaces and D-metric spaces were introduced by Gahler [5]and Dhage [3],respectively. The theory was further extended by Mustafa and Sims[6] to G-metric spaces. Novel concepts of D^* and S-metric spaces has been introduced by Shaban Sedghi[9,10]. In this paper, we find some new results of rectangular S-metric spaces and prove common fixed point theorems on some spaces .

2 PRELIMINARIES

Definition 2.1. [12] Let X be a non-empty set and $S: X^3 \to R^+, a$ function that satisfies the following properties:

i. $S(p,q,r) = 0$ if and only if p=q=r
ii. S(p,q,r)≤ $S(p,p,a) + S(q,q,a) + S(r,r,a)$

for all $a,p,q,r \in X$ (rectangle in equality)

Definition 2.2. [1] Let X be a non empty set and $S: X^3 \to R^+, a$ function satisfying the following properties:

i. $S(p,q,r) = 0$ if and only if p=q=r
ii. S(p,q,r)≤ $S(p,p,a) + S(q,q,a) + S(r,r,a)$

for all $p,q,r \in X$ and all distinct points $a \in X - \{p,q,r\}$. Then (X,S) is called a rectangular S- metric space .

Definition 2.3. [12] let (X,S) be an S-metric space and $A \subset X$

i. A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, $that$ is for $every$ $\in > 0$ there exists $n_0, \in N$ such that $n \geq n_0, S(x_n,x_n,x) < \in$. This case ,we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.
ii. A sequence $\{x_n\}$ is said to be Cauchy sequence if for each $\in > 0$, there exists $n_0 \in N$ such that $S(x_n,x_n,x_m) < \in$ for each $n,m \geq n_0$.
iii. The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent

Definition 2.4. [2] let f and g be weakly compatible self -maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$ then w is the unique common fixed point of f and g.

Lemma 2.5.[12] If (X,S) is an S-metric space, then we have $S(x,x,y) = S(y,y,x) for\ all\ x,y \in X$.

Lemma 2.6.[12] let (X,S) be an S-metric space. If $\{x_n\} and$ $\{y_n\}$ are sequences in X converging to x and y Respectively, that is ,$x_n \to x$ and $y_n \to y$ as $n \to \infty$, $then$ $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.

Lemma 2.7.[12] let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x ,then the limit x is unique .

Lemma 2.8.[12] let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.

Example 2.9. let $X = N \cup \{0\} and$ define $S: X \times X \times X \to R^+ \cup \{0\}$ by

$$S(p,q,r) = \begin{cases} 0 \; if \; p = q = r \\ px + qy + rz \; if \; p \neq q \neq r \end{cases}$$

Then (X,S) is a rectangular S- metric space.

3. MAIN RESULT

Theorem 3.1:let (X,S) be a complete rectangular S-metric space and let g and h be a self mapping on X. Assume that g and h satisfies the following condition :

$S(gx, gx, gy) \leq k[S(gx, gx, hy) + S(gy, gy, hx)]$

Where $0 \leq k < 0.1, also,$

i. $g(X) \subseteq h(X)$

ii. If h(X) is complete

Then g and h have a unique coincidence point in X.

Proof: let x_0 be any point in X.

Since $gx_0 \subseteq g(X) and \; g(X) \subseteq h(X)$

There exists a point x_1 in X such that $gx_0 = gx_1. As \; x_1 \in X$, it follows that $gx_1 \subseteq g(X)$.

Thus , we can select x_2 in X such that $gx_1 = hx_2$. Repeating this process iteratively we construct a sequence $\{x_n\}$ in X where $x_{n+1} \in X$, satisfies

$Gx_n = hx_{n+1} \; for \; all \; n.$

Now , consider

$$S(hx_{n+1}, hx_{n+1}, hx_n) = S(gx_n, gx_n, gx_{n-1})$$
$$\leq k[S(gx_n, gx_n, hx_{n-1}) + S(gx_{n-1}, gx_{n-1}, hx_n)]$$
$$= k[S(hx_{n+1}, hx_{n+1}, hx_{n-1}) + S(hx_n, hx_n, hx_n)]$$
$$\leq k[S(hx_{n+1}, hx_{n+1}, hx_n) + S(hx_{n+1}, hx_{n+1}, hx_n) + S(hx_{n-1}, hx_{n-1}, hx_n)]$$
$$= k[2S(hx_{n+1}, hx_{n+1}, hx_n) + S(hx_{n-1}, hx_{n-1}, hx_n)]$$
$$\Longrightarrow (1-2k)S(hx_{n+1}, hx_{n+1}, hx_n) \leq kS(hx_{n-1}, hx_{n-1}, hx_n)$$
$$\Longrightarrow S(hx_{n+1}, hx_{n+1}, hx_n) \leq \frac{k}{1-2k} S(hx_{n-1}, hx_{n-1}, hx_n)$$
$$= tS(hx_n, hx_n, hx_{n-1})$$
$$where \; t = \frac{k}{1-2k}$$
$$S(hx_{n+1}, hx_{n+1}, hx_n) \leq t^n S(hx_1, hx_1, hx_0)$$

For $m, n \in N$ and some $N \in N$ with $m < n$, we have

$$S(hx_n, hx_n, hx_m) \leq S(hx_n, hx_n, hx_{n-1}) + S(hx_n, hx_n, hx_{n-1}) + S(hx_m, hx_m, hx_{n-1})$$
$$\leq t^{n-1}S(hx_1, hx_1, hx_0) + t^{n-1}S(hx_1, hx_1, hx_0) + S(hx_m, hx_m, hx_{n-1})$$
$$= 2t^{n-1}S(hx_1, hx_1, hx_0) + S(hx_m, hx_m, hx_{n-1})$$
$$\leq 2t^{n-1}z + [S(hx_m, hx_m, hx_{n-2}) + S(hx_m, hx_m, hx_{n-2}) + S(hx_{n-1}, hx_{n-1}, hx_{n-2})$$

Where $z = S(hx_1, hx_1, hx_0)$

$$\leq 2t^{n-1}z + [2S(hx_m, hx_m, hx_{n-2}) + t^{n-2}S(hx_1, hx_1, hx_0)$$
$$= 2t^{n-1}z + t^{n-2}z + 2S(hx_m, hx_m, hx_{n-2})$$
$$\leq 2t^{n-1}z + t^{n-2}z + 2[2S(hx_m, hx_m, hx_{n-3}) + S(hx_{n-2}, hx_{n-2}, hx_{n-3})]$$
$$\leq 2t^{n-1}z + t^{n-2}z + 4S(hx_m, hx_m, hx_{n-3}) + 2t^{n-3}z$$
$$\leq 2t^{n-1}z + t^{n-2}z + 2t^{n-3}z + 4[2S(hx_m, hx_m, hx_{n-4}) + S(hx_{n-3}, hx_{n-3}, hx_{n-4})]$$
$$\leq 2t^{n-1}z + t^{n-2}z + 2t^{n-3}z + 8S[(hx_m, hx_m, hx_{n-4})] + 4t^{n-4}z$$
$$= 2t^{n-1}z + t^{n-2}z + 2t^{n-3}z + 4t^{n-4}z + \cdots$$
$$= 2t^{n-1}z + t^{n-2}z\left(1 + \frac{2}{t} + \frac{4}{t^2} + \frac{8}{t^3} + \cdots\right)$$
$$2t^{n-1}z + t^{n-2}z(1 - \frac{2}{t})^{-1}$$

Where $z = S(hx_1, hx_1, hx_0)$, as $n \to \infty$

And since $t < 1$ we have

$$\lim_{n \to \infty} S(hx_n, hx_n, hx_m) = 0$$

Since $\{hx_n\}$ is a Cauchy sequence in h(X). Since h(X) is complete , there exist q in h(X) such that $hx_n \to q, as \; n \to \infty$.

Consequently , we can find p in X such that $hp = q$.

$$S(gx_n, gx_n, gp) \leq k[S(gx_n, gx_n, hp) + S(gp, gp, hx_n)]$$
$$= k[S(hx_{n+1}, hx_{n+1}, hp) + S(gp, gp, hx_n)$$

$$= k[S(q,q,q) + S(gp,gp,q)]$$

Letting $n \to \infty$

$$S(q,q,gp) \leq kS(q,q,gp)$$

Since $S(q,q,q) = 0$ and $k < 1$, this is true if $S(q,q,gp)$

$$\Rightarrow gp = q$$

Therefore $gp = hp = q$

Hence q is the point of coincidence of g and h.

Now we show that g and h have a unique point of coincidence .

For this assume that there exists another point $u \in X$ such that $gu = hu = u$

Now

$$S(q,q,u) = S(gp,gp,gu)$$
$$\leq k[S(gp,gp,hu) + S(gu,gu,hp)]$$
$$= k[S(q,q,u) + S(u,u,q)]$$
$$\leq k[S(q,q,u) + S(q,q,u)]$$
$$= 2kS(q,q,u)$$

Since $k \in [0,0.1)$. It is true if $S(q,q,u) = 0$. So $q = u$.

Hence it is proved that g and h have a unique point of coincidence.

Since g and h are weakly compatible. So by proposition 1.4, g and h have a unique common fixed point in X.

4 CONCLUSION

- From this paper we conclude the establishment of common fixed point theorems.
- Generalization of S -metric spaces.
- Integration and extension of results.
- These conclusions of the paper in the context of fixed point theory in the rectangular S-metric spaces.

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