International Journal of Mathematics and Computer Research
ISSN: 2320-7167
Volume 13 Issue 10 October 2025, Page no. – 5744-5770
Index Copernicus ICV: 57.55, Impact Factor: 8.615
DOI: 10.47191/ijmcr/v13i10.09

Neimark–Sacker bifurcation and period-doubling bifurcation for a discretized fractional-order predator-prey system with Holling type II functional response

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Abstract: In this study, we examine the dynamical behaviors of the discrete form of a fractional-order predator-prey model with a Holling type II functional response. Specifically, we analyze the stability of its fixed points and the potential for various bifurcations by employing stability analysis methods and bifurcation theory. We explicitly show that, under specific parameter conditions, the system undergoes both a Neimark–Sacker bifurcation and a period-doubling bifurcation. Finally, several numerical simulations are performed to verify the analytical results derived earlier.

Keywords: Caputo fractional derivative, Discrete predator-prey system, Neimark–Sacker bifurcation, Period-doubling bifurcation.
Mathematics Subject Classification: 39A28, 39A30.

1. Introduction

In the 1920s, Alfred J. Lotka and Vito Volterra laid the groundwork for the modern study of predator-prey systems, establishing a pivotal theoretical basis for understanding their dynamics. The Lotka-Volterra model, which depicts the interplay between predator and prey populations, swiftly emerged as a central focus in dynamical research. To this day, investigating the intricate relationships between predators and their preys remains a prominent and evolving field of study. [1, 2].

Huang and Ruan [3] investigated the following classical Gause-type predator-prey model:

$$\begin{cases} \dot{x} = xg(x,k) - yp(x), \\ \dot{y} = y(-d + cq(x)), \end{cases} \tag{1.1}$$

here $x(t)$ and $y(t)$ represent the prey and predator densities respectively; the parameters c and d denote the efficiency with which predators convert consumed prey into their own growth and the predator mortality rate, in that order; $g(x,k)$ characterizes the specific growth rate of the prey when predators are absent. The predator's functional response to prey, denoted as $p(x)$, describes the change in the density of prey attacked by a single predator per unit time as the prey density varies. Additionally, the function $q(x)$ depicts the manner in which predators transform consumed prey into their own growth. The authors of [3] listed four kinds of functional response $p(x) = mx, \frac{mx}{a+x}, \frac{mx^2}{ax^2+bx+1}, \frac{mx}{ax^2+bx+1}$, for modeling the phenomena of predation. In cases where $q(x) = p(x)$, the predator-prey biological model(1.1) incorporating any of the four aforementioned response function types has been subjected to extensive research. For instance, relevant discussions can be found in the cited papers [4, 5, 6, 7] and their included references.

In the present study, we shall further examine the predator-prey model featuring the Holling-II functional response raised by Wu and Jiao [8] in the case of $p(N) = q(N) = \frac{\alpha N}{1+\alpha h N}$, i.e.

$$\begin{cases} \frac{dN}{dT} = Ng(N) - \frac{\alpha NP}{1+\alpha hN}, \\ \frac{dP}{dT} = \frac{c\alpha NP}{1+\alpha hN} - mP, \end{cases} \tag{1.2}$$

where the average growth rate of a typical prey species is assumed to be a logistic model $g(N) = R(1 - \frac{N}{k})$; N and P represent the population densities of the prey and predator at time T, in that order; the parameters k, α, and m correspond to the environmental carrying capacity of the prey in the absence of predators, the prey capture rate, and the intrinsic mortality rate of the predator, respectively; and c indicates the efficiency with which ingested prey are converted into predator biomass.

After simplyfing (1.2),we can now study the dynamical behavior of the following predator-prey system with Holling type II functional response:

$$\begin{cases} \frac{du(t)}{dt} = ru\left(1 - \frac{u}{k}\right) - \frac{\theta uv}{\delta+u}, \\ \frac{dv(t)}{dt} = \frac{\theta_1 uv}{\delta+u} - av. \end{cases} \tag{1.3}$$

The framework can be described as follows: In the absence of predator, the prey population grows in a logistic manner, governed by its intrinsic expansion rate r and the ecosystem's carrying capacity, k. However, the presence of predator leads to a reduction in the prey population by a factor of $\frac{\theta uv}{\delta+u}$, where θ represents the maximum predation rate, and δ is the half-saturation constant. Besides, the predation process follows the Holling type-II functional response, which accounts for the time predator requires to capture and process their prey. The consumption of prey by predator contributes to an increase in the predator population, which is determined by the ratio $\frac{\theta_1 uv}{\delta+u}$. Here, θ_1 denotes the rate at which consumed prey is converted into predator births, with the constraint $\theta_1 \leq \theta$.

Additionally, natural predator mortality occurs at a rate of a. Notably, the right-hand side of the second equation in [9] remains non-positive when $\theta_1 < a$. Therefore, within the framework of [9], we assume that $\theta_1 > a$ to ensure a biologically meaningful interpretation.

The notion of fractional derivatives dates back to the 18th century, with Liouville [10] being the first mathematician to put forward this concept. During the 20th century, mathematician Riesz first alluded to the concept of fractional derivatives and carried out studies on its properties in relevant literature. In [11], combine the studies of Liouville and Riesz to establish the Riesz–Liouville definition of fractional derivative that has been used up today. Subsequently, the mathematician Caputo introduced the Caputo definition of fractional derivative as follow [12].

Definition 1.1. *Denote*

$$ {}_0^cD_t^\alpha f(t) = J^{l-\alpha} f^{(l)}(t), \alpha > 0, $$

where $f^{(l)}$ denotes the derivative of f with order l, l is the nearest integer value of α, and J^q is the operator of the Riemann–Liouville integral of q order:

$$ J^q h(t) = \frac{\int_0^t (t-\tau_e)^{q-1} h(\tau_e) d\tau_e}{\Gamma(q)}, $$

where $\Gamma(q)$ is Euler's Gamma function, defined by $\Gamma(q) = \int_0^{+\infty} x^{q-1} e^{-x} dx$ with $Re\{q\} > 0$. The alpha-order Caputo differential operator is the term used to describe the operator ${}_0^cD_t^\alpha$.

Furthermore, fractional-order differential equations have attracted considerable attention and recognition because of their capacity to precisely characterize various nonlinear phenomena. Additionally, the construction of models grounded in fractional-order differential equations has become increasingly prevalent in the study of dynamical systems [13, 14, 15, 16]. Recently, a growing number of researchers have started to explore fractional-order biological models [17, 18, 19, 9, 20]. A key reason for this lies in the fact that fractional-order differential equations inherently correspond to systems with memory, a characteristic inherent to most biological systems, and are closely linked to fractals, which are prevalent in biological contexts. Due to the limited theories for analyzing the dynamics of fractional-order system, the stability study of fractional-order predator-prey system is only the beginning.

From a biological perspective, considering a fractional-order predator–prey system makes logical sense. In fractional calculus, the rate of change at any given moment, i.e., the fractional-order derivative, depends on the population density over a specific time interval. Consequently, fractional-order predator–prey systems have distinctive advantages in depicting the memory effect within populations.

In 2013, Ahmed [21] considered the following fractional-order predator-prey system:

$$\begin{cases} {}_0^C D_t^{\alpha} x(t) = x(t)\big(a - b\,x(t) - c\,y(t)\big) - h_1\,x(t), \\ {}_0^C D_t^{\alpha} y(t) = y(t)\big(-d + e\,x(t)\big) - h_2\,y(t), \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, ${}_0^C D_t^q$ is the fractional derivative in the sense of Caputo, x and y represent prey and predator densities, respectively, and all constants a, b, c, d, e, h_1 and h_2 are positive.

From a biological perspective, incorporating a fractional-order predator–prey system is well justified. In fractional calculus, the rate of change at any given moment—as captured by the fractional-order derivative—depends not only on the current population density but also on its history over a certain time interval. This feature makes fractional-order predator–prey models especially effective for describing memory effects and long-term dependencies in population dynamics.

In the past two decades, due to the advantages of fractional derivatives in exploring the memory effects of various ecological systems, a large number of mathematicians have shifted their focus to the research of fractional-order ecological systems, and have discovered many interesting dynamical properties, which are documented in relevant references [22, 23, 24]. Currently, a relatively comprehensive research framework has been developed for mathematical models of integer-order ecosystems, whereas research on fractional-order ecosystems remains in its preliminary phase. Thus, the present paper aims to incorporate the Caputo fractional derivative into system (1.2) and extends it to a fractional-order ecosystem. We will adopt the Caputo definition of fractional derivative to examine how the refuge effect in a fractional-order ecosystem influences the system's dynamics. Accordingly, we present the following fractional-order predator–prey system incorporating Holling-II functional response:

$$\begin{cases} {}_0^c D_t^q u(t) = ru\left(1 - \frac{u}{k}\right) - \frac{\theta uv}{\delta + u}, \\ {}_0^c D_t^q v(t) = \frac{\theta_1 uv}{\delta + u} - av. \end{cases} \tag{1.5}$$

where the meanings of all parameters are detailed in Table 1. In terms of integrating the Caputo fractional differential equation into an ecosystem model, refer also to [25].

Table 1: Biological meanings of parameters in system (1.5).

Parameter	Meaning
u, v	Prey and predator population densities respectively
r	Prey's intrinsic grow rate
k	Environmental carrying capacity
θ	Maximum prey predation level
δ	Half saturation constant
θ_1	The pace at which prey biomass turns to predator birth
a	The predator's mortality rate
q	Order of fractional derivative

Presently, there is a lack of comprehensive dynamical analysis methods for continuous fractional-order predator-prey systems. Typically, it is not feasible to derive an exact solution for a complex differential equation or system. For example, in the reference [26], Analyses of fractional-order systems have primarily centered on the global asymptotic stability of the predator-extinction equilibrium point. For this reason, numerous researchers have resorted to computing approximate solutions via computer methods.

Computers operate based on discrete points, making it both natural and practical to discretize the corresponding continuous model. In the reference [27], The authors examined a variety of discretized predator-prey models and noted that, in comparison to their continuous counterparts, these discrete models exhibit a broader spectrum of dynamical behaviors and present certain advantages in numerical simulations. In the reference [28], The authors utilized the piecewise constant approximation (PCA) method to discretize a continuous fractional-order predator–prey system, analyzed its dynamical properties, and discussed the types of bifurcations exhibited therein. Their research inspires us to consider the discrete counterpart of system (1.5). Thus, we employ the PCA method here to discretize model (1.5), with the specific steps outlined as follows.

Suppose the initial conditions of system (1.5) are $u(0) = u_0$ and $v(0) = v_0$. For a given step size ρ, denote $u(n\rho) = u_n$ and $v(n\rho) = v_n$ for $n = 0, 1, 2, \cdots$. The PCA method is as follows:

$$\begin{cases} {}_0^C D_t^\alpha u(t) = ru\left(\rho\left[\frac{t}{\rho}\right]\right)\left(1 - \frac{u\left(\rho\left[\frac{t}{\rho}\right]\right)}{k}\right) - \frac{\theta u\left(\rho\left[\frac{t}{\rho}\right]\right)v\left(\rho\left[\frac{t}{\rho}\right]\right)}{\delta + u\left(\rho\left[\frac{t}{\rho}\right]\right)}, \\ {}_0^C D_t^\alpha v(t) = \frac{\theta_1 u\left(\rho\left[\frac{t}{\rho}\right]\right)v\left(\rho\left[\frac{t}{\rho}\right]\right)}{\delta + u\left(\rho\left[\frac{t}{\rho}\right]\right)} - av\left(\rho\left[\frac{t}{\rho}\right]\right). \end{cases}$$

First, let $t \in [0, \rho)$, then $\frac{t}{\rho} \in [0, 1)$. Thus, we obtain

$$\begin{cases} {}_0^c D_t^q u(t) = ru_0\left(1 - \frac{u_0}{k}\right) - \frac{\theta u_0 v_0}{\delta + u_0}, \\ {}_0^c D_t^q v(t) = \frac{\theta_1 u_0 v_0}{\delta + u_0} - av_0. \end{cases} \tag{1.6}$$

The system (1.6) may be changed into

$$\begin{aligned} u(t) &= u_0 + \mathcal{J}^\alpha \left(ru_0 \left(1 - \frac{u_0}{k}\right) - \frac{\theta_1 u_0 v_0}{\delta + u_0} \right) \\ &= u_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)} \left(ru_0 \left(1 - \frac{u_0}{k}\right) - \frac{\theta_1 u_0 v_0}{\delta + u_0} \right), \\ v(t) &= v_0 + \mathcal{J}^\alpha \left(\frac{\theta_1 u_0 v_0}{\delta + u_0} - av_0 \right) \\ &= v_0 + \frac{t^\alpha}{\alpha\Gamma(\alpha)} \left(\frac{\theta_1 u_0 v_0}{\delta + u_0} - av_0 \right). \end{aligned}$$

Second, let $t \in [\rho, 2\rho)$, so $\frac{t}{\rho} \in [1, 2)$. Then

$$\begin{cases} {}_0^cD_t^q u(t) = ru_1 \left(1 - \frac{u_1}{k}\right) - \frac{\theta u_1 v_1}{\delta + u_1}, \\ {}_0^cD_t^q v(t) = \frac{\theta_1 u_1 v_1}{\delta + u_1} - av_1. \end{cases} \tag{1.7}$$

Upon simplifying equation (1.7), the following solution can be derived

$$\begin{aligned} u(t) &= u_1 + \mathcal{J}_\rho^\alpha \left(ru_1 \left(1 - \frac{u_1}{k}\right) - \frac{\theta u_1 v_1}{\delta + u_1} \right) \\ &= u_1 + \frac{(t-\rho)^\alpha}{\alpha\Gamma(\alpha)} \left(ru_1 \left(1 - \frac{u_1}{k}\right) - \frac{\theta u_1 v_1}{\delta + u_1} \right), \\ v(t) &= v_1 + \mathcal{J}_\rho^\alpha \left(\frac{\theta_1 u_1 v_1}{\delta + u_1} - av_1 \right) \\ &= v_1 + \frac{(t-\rho)^\alpha}{\alpha\Gamma(\alpha)} \left(\frac{\theta_1 u_1 v_1}{\delta + u_1} - av_1 \right), \end{aligned}$$

where $\mathcal{J}_\rho^\alpha = \frac{1}{\Gamma(\alpha)} \int_\rho^t (t - \tau_e)^{\alpha-1} d\tau_e$, $0 < \alpha \leq 1$. After n iterations, the result obtained is as follows

$$\begin{aligned} u(t) &= u_n + \frac{(t-n\rho)^\alpha}{\alpha\Gamma(\alpha)} \left(ru_n \left(1 - \frac{u_n}{k}\right) - \frac{\theta u_n v_n}{\delta + u_n} \right), \\ v(t) &= v_n + \frac{(t-n\rho)^\alpha}{\alpha\Gamma(\alpha)} \left(\frac{\theta_1 u_n v_n}{\delta + u_n} - av_n \right), \end{aligned}$$

where $t \in [n\rho, (n+1)\rho)$. Letting $t \to ((n+1)\rho)^-$, the system above reads

$$\begin{cases} u_{n+1} = u_n + \frac{\rho^\alpha}{\Gamma(\alpha+1)} \left(ru_n \left(1 - \frac{u_n}{k}\right) - \frac{\theta u_n v_n}{\delta + u_n} \right), \\ v_{n+1} = v_n + \frac{\rho^\alpha}{\Gamma(\alpha+1)} \left(\frac{\theta_1 u_n v_n}{\delta + u_n} - av_n \right). \end{cases} \tag{1.8}$$

System (1.8) is our goal to considered in the paper and the meanings of the parameters in system (1.8) are the same as in Table 1.

The remaining structure of this paper is organized as follows: In Section 2, relevant preliminaries are presented, including definitions, lemmas, and theorems that will be applied to analyze the dynamical properties of system (1.8). Section 3 investigates the existence and stability of the fixed points of system (1.8) respectively. In Section 4, we demonstrate that under specific parameter conditions, system (1.8) undergoes a Neimark–Sacker bifurcation and a period-doubling bifurcation. Section 5 conducts numerical simulations to verify the findings of our theoretical analysis. Finally, in Section 6, significant conclusions are drawn based on the results obtained in the preceding sections.

2. The Existence and Stability of Fixed Points

In this section, we first determine the fixed points of system (1.8) and then examine their local stability.

It is clear that system (1.8) always has fixed points $E_0(0,0)$ and $E_1(k,0)$ for all parameter values. Aside from these two fixed points, system (1.8) has a unique coexistence fixed point $E_2(x^*, y^*)$ if $\theta_1 > \frac{a(\delta+k)}{k}$, where $x^* = \frac{a\delta}{\theta_1 - a}$, $y^* = \frac{r\delta\theta_1(k\theta_1 - a(k+\delta))}{k\theta(\theta_1-a)^2}$.

The Jacobian matrix of system (1.8) at any fixed point $E(u_n, v_n)$ takes the form:

$$J(E) = \begin{pmatrix} 1 + \frac{\rho^\alpha}{\Gamma(\alpha+1)}\left(r(1-\frac{2u_n}{k}) - \frac{\theta\delta v_n}{(\delta+u_n)^2}\right) & -\frac{\rho^\alpha}{\Gamma(\alpha+1)}\frac{\theta u_n}{\delta+u_n} \\ \frac{\rho^\alpha}{\Gamma(\alpha+1)}\frac{\theta_1\delta v_n}{(\delta+u_n)^2} & 1 + \frac{\rho^\alpha}{\Gamma(\alpha+1)}\left(\frac{\theta_1 u_n}{\delta+u_n} - a\right) \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix $J(E)$ reads

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$\begin{aligned} p &= 2 + \frac{\rho^\alpha}{\Gamma(\alpha+1)}\left[r\left(1-\frac{2u_n}{k}\right) - \frac{\theta\delta v_n}{(\delta+u_n)^2} + \frac{\theta_1 u_n}{\delta+u_n} - a\right], \\ q &= a_{11}a_{22} + \left(\frac{\rho^\alpha}{\Gamma(\alpha+1)}\right)^2 \cdot \frac{\theta\theta_1\delta u_n v_n}{(\delta+u_n)^3}, \\ a_{11} &= 1 + \frac{\rho^\alpha}{\Gamma(\alpha+1)}\left(r\left(1-\frac{2u_n}{k}\right) - \frac{\theta\delta v_n}{(\delta+u_n)^2}\right), \\ a_{22} &= 1 + \frac{\rho^\alpha}{\Gamma(\alpha+1)}\left(\frac{\theta_1 u_n}{\delta+u_n} - a\right). \end{aligned}$$

Now let $\Delta = \frac{\rho^\alpha}{\Gamma(\alpha+1)}$ and $\Omega = \frac{\theta_1 k}{\delta+k} - a$. The following results are obtained.

Theorem 2.1. *The following assertions regarding the trivial equilibrium point $E_0(0,0)$ of system (1.8) hold true.*

(1) If $\Delta > \frac{2}{a}$, then E_0 is an unstable node, i.e., a source;

(2) if $\Delta = \frac{2}{a}$, then E_0 is non-hyperbolic;

(3) if $\Delta < \frac{2}{a}$, then E_0 is a saddle.

Proof. Substituting the trivial equilibrium point E (0,0) into the Jacobian matrix $J(E)$, we obtain

$$J(E_0) = \begin{pmatrix} 1+\Delta r & 0 \\ 0 & 1-\Delta a \end{pmatrix}.$$

It is straightforward to observe that the Jacobian matrix $J(E_0)$ has two eigenvalues that satisfy the following condition: $|\lambda_1| = |1+\Delta r| > 1$ and $|\lambda_2| = |1-\Delta a| < (=,>)1$ for $\Delta < (=,>)\frac{2}{a}$. By using Definition 2.3, we can derive these conclusions in Theorem 3.1.

Theorem 2.2. *The following conclusions apply to the fixed point $E_1(k,0)$ of system (1.8).*

1. If $\theta_1 > \frac{(a+1)(\delta+k)}{k}$, then,

(1) for $\Delta > \frac{2}{r}$, E_1 is an unstable node, i.e., a source;

(2) for $\Delta = \frac{2}{r}$, E_1 is non-hyperbolic;

(3) for $\Delta < \frac{2}{r}$, E_1 is a saddle.

2. If $\theta_1 = \frac{(a+1)(\delta+k)}{k}$, then, E_1 is non-hyperbolic.

3. If $\theta_1 < \frac{(a+1)(\delta+k)}{k}$, consider the following three situations:

(1) $-\frac{2}{\Omega} < \frac{2}{r}$. Then,

(a) for $0 < \Delta < -\frac{2}{\Omega}$, E_1 is a saddle;

(b) for $\Delta = -\frac{2}{\Omega}$, E_1 is non-hyperbolic;

(c) for $-\frac{2}{\Omega} < \Delta < \frac{2}{r}$, E_1 is a stable node, i.e., a sink;

(d) for $\Delta = \frac{2}{r}$, E_1 is non-hyperbolic;

(e) for $\Delta > \frac{2}{r}$, E_1 is a saddle.

(2) $-\frac{2}{\Omega} = \frac{2}{r}$. Then,

(a) for $0 < \Delta < \frac{2}{r}$, E_1 is a saddle;

(b) for $\Delta = \frac{2}{r}$, E_1 is non-hyperbolic;

(c) for $\Delta > \frac{2}{r}$, E_1 is a saddle.

(3) $-\frac{2}{\Omega} > \frac{2}{r}$. Then,

(a) for $0 < \Delta < \frac{2}{r}$, E_1 is a saddle;

(b) for $\Delta = \frac{2}{r}$, E_1 is non-hyperbolic;

(c) for $\frac{2}{r} < \Delta < -\frac{2}{\Omega}$, E_1 is an unstable node, i.e., a source;

(d) for $\Delta = -\frac{2}{\Omega}$, E_1 is non-hyperbolic;

(e) for $\Delta > -\frac{2}{\Omega}$, E_1 is a saddle.

Proof. Substituting the trivial equilibrium point $E_1(k,0)$ into the Jacobian matrix $J(E)$, yields the following result

$$J(E_1) = \begin{pmatrix} 1-\Delta r & -\Delta\frac{\theta k}{\delta+k} \\ 0 & 1+\Delta(\frac{\theta_1 k}{\delta+k} - a) \end{pmatrix}.$$

Obviously, $\lambda_1 = 1-\Delta r$ and $\lambda_2 = 1+\Delta(\frac{\theta_1 k}{\delta+k} - a)$. Now consider the following three case.

Case 1: $\theta_1 > \frac{a(\delta+k)}{k}$. Then $\frac{\theta_1 k}{\delta+k} - a > 0$. So $|\lambda_2| > 1$. If $\Delta > \frac{2}{r}$, then $|\lambda_1| > 1$, so E_1 is an unsatble node, namely, a source; if $\Delta = \frac{2}{r}$, then $|\lambda_1| = 1$, so E_1 is non-hyperbolic; if $\Delta < \frac{2}{r}$, then $|\lambda_1| < 1$, so E_1 is a saddle.

Case 2: $\theta_1 = \frac{a(\delta+k)}{k}$. Then $\frac{\theta_1 k}{\delta+k} - a = 0$, so $|\lambda_2| = 1$, and hence E_1 is non-hyperbolic.

Case 3: $\theta_1 < \frac{a(\delta+k)}{k}$. Then $\Omega = \frac{\theta_1 k}{\delta+k} - a < 0$. Hence $\lambda_2 = 1 + \Delta\Omega < 1$. Consider the following three situations:

(1) $-\frac{2}{\Omega} < \frac{2}{r}$. We further consider the following five cases:

(a) $0 < \Delta < -\frac{2}{\Omega}$. Lemma 1 (i.3) reads $|\lambda_1| < 1$ and $|\lambda_2| > 1$, implying E_1 is a saddle.

(b) $\Delta = -\frac{2}{\Omega}$. Then $\lambda_2 = -1$. So E_1 is non-hyperbolic.

(c) $-\frac{2}{\Omega} < \Delta < \frac{2}{r}$. Then $|\lambda_1| < 1$, $|\lambda_2| < 1$. This indicates that E_1 is a stable node, i.e., a sink.

(d) $\Delta = \frac{2}{r}$. Then $\lambda_1 = -1$. So E_1 is non-hyperbolic.

(e) $\Delta > \frac{2}{r}$. Then $|\lambda_1| > 1$, $|\lambda_2| < 1$, reading that E_1 is a saddle.

(2) $-\frac{2}{\Omega} = \frac{2}{r}$. We further consider the following three cases:

(a) $0 < \Delta < \frac{2}{r}$. Then $|\lambda_1| < 1$, $|\lambda_2| > 1$. Therefore, E_1 is a saddle.

(b) $\Delta = \frac{2}{r}$. Then $\lambda_1 = -1$, which shows E_1 is non-hyperbolic.

(c) $\Delta > \frac{2}{r}$. Then $|\lambda_1| > 1$, $|\lambda_2| < 1$, that displays that E_1 is a saddle.

(3) $-\frac{2}{\Omega} > \frac{2}{r}$. We further consider the following five cases:

(a) $0 < \Delta < \frac{2}{r}$. Then $|\lambda_1| < 1$, $|\lambda_2| > 1$, saying E_1 is a saddle.

(b) $\Delta = \frac{2}{r}$. Then $\lambda_1 = -1$. So E_1 is non-hyperbolic.

(c) $\frac{2}{r} < \Delta < -\frac{2}{\Omega}$. Then $|\lambda_1| > 1$, $|\lambda_2| > 1$, showing E_1 is an unstable node, i.e., a source.

(d) $\Delta = -\frac{2}{\Omega}$. Then $\lambda_2 = -1$. Hence E_1 is non-hyperbolic.

(e) $\Delta > -\frac{2}{\Omega}$. Then $|\lambda_1| > 1$, $|\lambda_2| < 1$. That is to say, E_1 is a saddle.

The proof is over.

Theorem 2.3. *Let* $A_1 = \dfrac{\theta\theta_1\delta v^*}{(\delta+u^*)^2}$ *and* $A_2 = r - \dfrac{2ru^*}{k} - \dfrac{r\delta(1-\frac{u^*}{k})}{\delta+u^*}$*. When* $a < \frac{k\theta_1}{k+\delta}$, $E_2 = (\frac{a\delta}{\theta_1 - a}, \frac{r\delta\theta_1(k\theta_1 - a(k+\delta))}{k\theta(\theta_1-a)^2})$ *is a positive fixed point of system (1.8). Then the results summarized in Tables 2 are valid for the positive fixed points* E_2 *of system (1.8) .*

Proof. The Jacobian matrix of system (1.8) at the fixed point E_2 may be simplified into

$$\mathbf{J}(E_2) = \begin{pmatrix} 1+\Delta A_2 & -\Delta\theta \\ \frac{\Delta A_1}{\theta} & 1 \end{pmatrix}$$

From this, we derive the characteristic polynomial of the Jacobian matrix $J(E_2)$

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$p = \ 2 + \Delta A_2,$$
$$q = 1 + \Delta A_2 + \Delta^2 A_1.$$

It is clear that

$$F(1) = \Delta^2 A_1 > 0 \quad and$$
$$F(-1) = 4 + 2\Delta A_2 + \Delta^2 A_1.$$

Note also that $q > (=, <)\, 1 \Leftrightarrow \Delta > (=, <) - \frac{A_2}{A_1}$. Now, consider the following three cases:

Case 1: $A_2^2 - 4A_1 < 0$.

This implies that $F(-1) > 0$. Now, we consider the following three subcases:

Subcase 1: $\Delta < -\frac{A_2}{A_1}$.

Then, $q < 1$. Lemma 1 reads $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Thus, E_2 is a stable node, i.e., a sink.

Subcase 2: $\Delta = -\frac{A_2}{A_1}$.

Then, $q = 1, -2 < p < 2$, so E_2 is non-hyperbolic.

Subcase 3: $\Delta > -\frac{A_2}{A_1}$.

Then, $q > 1$. Thus, $|\lambda_1| > 1$, $|\lambda_2| > 1$, and hence E_2 is an unstable node, i.e., a source.

Case 2: $A_2^2 - 4A_1 = 0$.

Consider the following three subcases:

Subcase 1: $\Delta < -\frac{A_2}{A_1}$.

Then, $F(-1) > 0$, $q < 1$. Hence, $|\lambda_1| < 1$, $|\lambda_2| < 1$, and so E_2 is a stable node, i.e., a sink.

Subcase 2: $\Delta = -\frac{A_2}{A_1}$.

Then, $F(-1) = 0$. Thus, E_2 is non-hyperbolic.

Subcase 3: $\Delta > -\frac{A_2}{A_1}$.

Then, $F(-1) > 0$ and $q > 1$. It follows from Lemma 1 (i4) that $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Thus, E_2 is an unstable node, i.e., a source.

Case 3: $A_2^2 - 4A_1 > 0$.

Then, $A_2 > 2\sqrt{A_1}$ or $A_2 < -2\sqrt{A_1}$. Consider the following two subcases:

Subcase 1: $A_2 > 2\sqrt{A_1}$.

Then, $\Delta > 0 > \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1} > -\frac{A_2}{A_1}$. So, $F(-1) > 0$, $q > 1$, which, in view of Lemma 1 (i.4), implies $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Thus, E_2 is an unstable node, i.e., a source.

Subcase 2: $A_2 < -2\sqrt{A_1}$.

We further consider the following five subsubcases:

Subsubcase 1: $0 < \Delta < \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1} < \frac{A_2}{A_1}$.

Then, $F(-1) > 0, q < 1$, indicating $|\lambda_1| < 1$, $|\lambda_2| < 1$. Hence, E_2 is a stable node, i.e., a sink.

Subsubcase 2: $\Delta = \frac{-A_2-\sqrt{A_2^2-4A_1}}{A_1}$.

Then, $F(-1) = 0$ and so E_2 is non-hyperbolic.

Subsubcase 3: $\frac{-A_2-\sqrt{A_2^2-4A_1}}{A_1} < \Delta < \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1}$.

Then, $F(-1) < 0$. In light of Lemma 1 (i.3), E_2 is a saddle.

Subsubcase 4: $\Delta = \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1}$.

Then, $F(-1) = 0$ and so E_2 is non-hyperbolic.

Subsubcase 5: $\Delta > \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1} > \frac{A_2}{A_1}$.

Then, $F(-1) > 0$, $q > 1$ and so $|\lambda_1| > 1$, $|\lambda_2| > 1$. Therefore, E_2 is an unstable node, i.e., a source.

The proof is finished.

Table 2. Properties of the fixed point E_2 of system (1.8).

Conditions			Eigenvalues	Properties
$A_2^2 - 4A < 0$	$\Delta < -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert < 1,\ \lvert\lambda_2\rvert < 1$	stable node
	$\Delta = -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert = 1$ or $\lvert\lambda_2\rvert = 1$	non-hyperbolic
	$\Delta > -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert > 1,\ \lvert\lambda_2\rvert > 1$	unstable node
$A_2^2 - 4A_1 = 0$	$\Delta < -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert < 1,\ \lvert\lambda_2\rvert < 1$	stable node
	$\Delta = -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert = 1$ or $\lvert\lambda_2\rvert = 1$	non-hyperbolic
	$\Delta > -\frac{A_2}{A_1}$		$\lvert\lambda_1\rvert > 1,\ \lvert\lambda_2\rvert > 1$	unstable node
$A_2^2 - 4A_1 > 0$	$A_2 > 2\sqrt{A_1}$		$\lvert\lambda_1\rvert > 1,\ \lvert\lambda_2\rvert > 1$	unstable node
	$A_2 < -2\sqrt{A_1}$	$\Delta < \frac{-A_2-\sqrt{A_2^2-4A_1}}{A_1}$	$\lvert\lambda_1\rvert < 1,\ \lvert\lambda_2\rvert < 1$	stable node
		$\Delta = \frac{-A_2-\sqrt{A_2^2-4A_1}}{A_1}$	$\lvert\lambda_1\rvert = (\neq)1,\ \lvert\lambda_2\rvert \neq (=)1$	non-hyperbolic
		$\frac{-A_2-\sqrt{A_2^2-4A_1}}{A_1} < \Delta < \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1}$	$\lvert\lambda_1\rvert < (>)1,\ \lvert\lambda_2\rvert > (<)1$	saddle
		$\Delta = \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1}$	$\lvert\lambda_1\rvert \neq (=)1,\ \lvert\lambda_2\rvert = (\neq)1$	non-hyperbolic
		$\Delta > \frac{-A_2+\sqrt{A_2^2-4A_1}}{A_1}$	$\lvert\lambda_1\rvert > 1,\ \lvert\lambda_2\rvert > 1$	unstable node

3. Bifurcation Analysis

In this section, we primarily utilize the Center Manifold Theorem and local bifurcation theory to analyze the local bifurcation behavior of system (1.8) at the fixed point E_2, taking into account its practical biological significance.

3.1. Neimark–Sacker Bifurcation at the Fixed Point E_2

From Case 1 in the proof of Theorem 3.3 for the stability of the positive fixed point E_2,it can be observed that the dimensions of both the stable manifold and the unstable manifold of system (1.8) at the positive fixed point E_2 undergo changes when Δ varies in the vicinity of Δ_0 (correspondingly, ρ varies in the vicinity of ρ_0) for $A < 4\Omega$, where

$$\Delta_0 = \frac{A_2}{A_1}, \quad \rho_0 = (\Gamma(\alpha+1)\Delta_0)^{\frac{1}{\alpha}} \tag{3.1}$$

Thus, a bifurcation may arise, which will later be shown to be a Neimark–Sacker bifurcation. Let

$$S = \{(\theta, \theta_1, \delta, k, h, r, a, q, \alpha, \rho) \in R_+^{10} | a < \frac{k\theta_1}{k+\delta}, A_2^2 - 4A_1 < 0\}.$$

To investigate the Neimark–Sacker bifurcation, the following steps are carried out.

Let $x_n = u_n - x^*$, $y_n = v_n - y^*$, which transforms the fixed point E_2 to the origin $O(0,0)$. Assume that ρ^* is a small perturbation of ρ, i.e., $\rho^* = \rho - \rho_0$, with $0 < |\rho^*| \ll 1$. After undergoing translation and perturbation, system (1.8) assumes the following form:

$$\begin{cases} x_{n+1} = x_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(r(x_n + x^*) \left(1 - \frac{x_n+x^*}{k}\right) - \frac{\theta(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} \right), \\ y_{n+1} = y_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(\frac{\theta_1(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} - a(y_n + y^*) \right). \end{cases} \tag{3.2}$$

Expanding system (3.2) around $O(0,0)$ using the Taylor series up to the third order yields the following system::

$$\begin{cases} x_{n+1} = \epsilon_{10}x_n + \epsilon_{01}y_n + \epsilon_{20}x_n^2 + \epsilon_{11}x_ny_n + \epsilon_{02}y_n^2 \\ \qquad\quad + \epsilon_{30}x_n^3 + \epsilon_{21}x_n^2y_n + \epsilon_{12}x_ny_n^2 + \epsilon_{03}y_n^3 + o(\rho_1^3), \\ y_{n+1} = \zeta_{10}x_n + \zeta_{01}y_n + \zeta_{20}x_n^2 + \zeta_{11}x_ny_n + \zeta_{02}y_n^2 \\ \qquad\quad + \zeta_{30}x_n^3 + \zeta_{21}x_n^2y_n + \zeta_{12}x_ny_n^2 + \zeta_{03}y_n^3 + o(\rho_1^3), \end{cases} \tag{3.3}$$

where $\rho_1 = \sqrt{x_n^2 + y_n^2}$,

$$\epsilon_{10} = 1 + \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\left[r\left(1 - \frac{x^*}{k}\right) - \frac{\theta y^*}{\delta + x^*}\right], \epsilon_{01} = -\frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta x^*}{\delta + x^*},$$
$$\epsilon_{20} = -\frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\left[\frac{r}{k} + \frac{\theta y^*}{(\delta + x^*)^2}\right], \epsilon_{02} = 0, \epsilon_{11} = -\frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta}{\delta + x^*},$$
$$\epsilon_{30} = -\frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\left(\frac{2r}{k^2} + \frac{2\theta y^*}{(\delta + x^*)^3}\right), \epsilon_{03} = 0, \epsilon_{21} = -\frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta}{(\delta + x^*)^2}, \epsilon_{12} = 0,$$
$$\zeta_{10} = \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta_1 y^*}{\delta + x^*}, \zeta_{01} = 1 + \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\left[\frac{\theta_1 x^*}{\delta + x^*} - a\right], \zeta_{20} = 0,$$
$$\zeta_{02} = 0, \zeta_{11} = \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta_1}{\delta + x^*}, \zeta_{30} = \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{2\theta_1 y^*}{(\delta + x^*)^3},$$
$$\zeta_{03} = 0, \zeta_{21} = \frac{(\rho^* + \rho_0)^\alpha}{\Gamma(\alpha+1)}\frac{\theta_1}{(\delta + x^*)^2}, \zeta_{12} = 0.$$

The characteristic equation of the linearized equation of system (3.3) is

$$F(\lambda) = \lambda^2 - p(\rho^*)\lambda + q(\rho^*) = 0,$$

where $p(\rho^*) = 2 - A\Delta(\rho^*)$, $q(\rho^*) = 1 - A\Delta(\rho^*) + A\Omega\Delta^2(\rho^*)$, $\Delta(\rho^*) = \frac{(\rho_0+\rho^*)^\alpha}{\Gamma(\alpha+1)}$. Noting that $\Delta(0) = \Delta_0$ and $p^2(0) - 4q(0) = \Delta_0^2(A^2 - 4A\Omega) < 0$, the two roots of the characteristic equation are

$$\lambda_{1,2}(\rho^*) = \frac{p(\rho^*) \pm i\sqrt{4q(\rho^*) - p^2(\rho^*)}}{2}.$$

Moreover,

$$\left(|\lambda_{1,2}(\rho^*)|\right)\Big|_{\rho^*=0} = \sqrt{q(\rho^*)}\Big|_{\rho^*=0} = 1,$$
$$\left(\frac{d|\lambda_{1,2}(\rho^*)|}{d\rho^*}\right)\Big|_{\rho^*=0} = \frac{\alpha\rho_0^{\alpha-1}A}{2\Gamma(\alpha+1)} \neq 0.$$

It is evident that $\lambda_{1,2}^i(0) \neq 1$, for $i = 1, 2, 3, 4$. Thus, the transversality and non-degeneracy conditions required for the occurrence of a Neimark–Sacker bifurcation are satisfied.

In order to derive the normal form of system (3.3), let

$$T = \begin{pmatrix} 0 & \epsilon_{01} \\ \mu & 1-\omega \end{pmatrix},$$

where $\omega = -\frac{p(\rho^*)}{2}$, $\mu = \frac{\sqrt{4q(\rho^*)-p^2(\rho^*)}}{2}$. Then, we have

$$T^{-1} = \begin{pmatrix} \frac{\omega-1}{\mu\epsilon_{01}} & \frac{1}{\mu} \\ \frac{1}{\epsilon_{01}} & 0 \end{pmatrix}.$$

Transform the variables x and y to

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$$(x, y)^T = T(U, V)^T,$$

then, system (3.3) becomes to the following form:

$$\begin{pmatrix} U \\ V \end{pmatrix} \to \begin{pmatrix} \omega & -\mu \\ \mu & \omega \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} F(U,V) + o(\rho_2^3) \\ G(U,V) + o(\rho_2^3) \end{pmatrix}, \tag{3.4}$$

where $\rho_2 = \sqrt{u^2 + v^2}$,

$$F(U,V) = c_{20}U^2 + c_{11}UV + c_{02}V^2 + c_{30}U^3 + c_{21}U^2V + c_{12}UV^2 + c_{03}V^3,$$

$$G(U,V) = d_{20}U^2 + d_{11}UV + d_{02}V^2 + d_{30}U^3 + d_{21}U^2V + d_{12}UV^2 + d_{03}V^3,$$

with $u = \frac{(\omega-1)x}{\mu\epsilon_{01}} + \frac{y}{\mu}$, and $v = \frac{x}{\epsilon_{01}}$,

$$c_{20} = \frac{\epsilon_{20}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{20}}{\mu}, \quad c_{11} = \frac{\epsilon_{11}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{11}}{\mu}, \quad c_{02} = \frac{\epsilon_{02}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{02}}{\mu},$$

$$c_{30} = \frac{\epsilon_{30}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{30}}{\mu}, \quad c_{21} = \frac{\epsilon_{21}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{21}}{\mu}, \quad c_{12} = \frac{\epsilon_{12}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{12}}{\mu},$$

$$c_{03} = \frac{\epsilon_{03}(\omega-1)}{\mu\epsilon_{01}} + \frac{\zeta_{03}}{\mu}, \quad d_{20} = \frac{\epsilon_{20}}{\epsilon_{01}}, \quad d_{11} = \frac{\epsilon_{11}}{\epsilon_{01}}, \quad d_{02} = \frac{\epsilon_{02}}{\epsilon_{01}}, \quad d_{30} = \frac{\epsilon_{30}}{\epsilon_{01}},$$

$$d_{21} = \frac{\epsilon_{21}}{\epsilon_{01}}, \quad d_{12} = \frac{\epsilon_{12}}{\epsilon_{01}}, \quad d_{03} = \frac{\epsilon_{03}}{\epsilon_{01}}.$$

Moreover,

$$\begin{aligned}
F_{UU}|_{(0,0)} &= 2c_{02}\mu^3, \quad F_{UV}|_{(0,0)} = c_{11}\epsilon_{01}\mu + 2c_{02}\mu(1-\omega), \\
F_{VV}|_{(0,0)} &= 2c_{20}\epsilon_{01}^2 + 2c_{11}\epsilon_{01}(1-\omega), \quad F_{UUU}|_{(0,0)} = 6c_{03}\mu^3, \\
F_{UUV}|_{(0,0)} &= 2c_{21}\epsilon_{01}\mu^2 + 6c_{03}\mu^2(1-\omega), \\
F_{UVV}|_{(0,0)} &= 2c_{21}\epsilon_{01}^2\mu + 4c_{12}\epsilon_{01}\mu(1-\omega) + 6c_{03}\mu(1-\omega)^2, \\
F_{VVV}|_{(0,0)} &= 4(1-\omega)^3 + 6c_{30}\epsilon_{01}^3 + 4c_{21}\epsilon_{01}^2(1-\omega) + 6c_{12}\epsilon_{01}(1-\omega)^2, \\
G_{UU}|_{(0,0)} &= 2d_{02}\mu^3, \quad G_{UV}|_{(0,0)} = d_{11}\epsilon_{01}\mu + 2d_{02}\mu(1-\omega), \\
G_{VV}|_{(0,0)} &= 2d_{20}\epsilon_{01}^2 + 2d_{11}\epsilon_{01}(1-\omega), \quad G_{UUU}|_{(0,0)} = 6c_{03}\mu^3, \\
G_{UUV}|_{(0,0)} &= 2d_{21}\epsilon_{01}\mu^2 + 6d_{03}\mu^2(1-\omega), \\
G_{UVV}|_{(0,0)} &= 2d_{21}\epsilon_{01}^2\mu + 4d_{12}\epsilon_{01}\mu(1-\omega) + 6d_{03}\mu(1-\omega)^2, \\
G_{VVV}|_{(0,0)} &= 4(1-\omega)^3 + 6d_{30}\epsilon_{01}^3 + 4d_{21}\epsilon_{01}^2(1-\omega) + 6d_{12}\epsilon_{01}(1-\omega)^2.
\end{aligned}$$

To confirm that system (1.8) undergoes a Neimark-Sacker bifurcation and to determine the stability and direction of the bifurcation curve, the discriminant L needs to be computed and must not be zero, where

$$L = -\mathrm{Re}\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2 + \mathrm{Re}(\lambda_2\tau_{21}), \qquad (3.5)$$

$$\tau_{20} = \frac{1}{8}\left[F_{XX} - F_{YY} + 2G_{XY} + i(G_{XX} - G_{YY} - 2F_{XY})\right]|_{(0,0)},$$
$$\tau_{11} = \frac{1}{4}\left[F_{XX} + F_{YY} + i(G_{XX} + G_{YY})\right]|_{(0,0)},$$
$$\tau_{02} = \frac{1}{8}\left[F_{XX} - F_{YY} - 2G_{XY} + i(G_{XX} - G_{YY} + 2F_{XY})\right]|_{(0,0)},$$

$$\tau_{21} = \frac{1}{16}\left[F_{XXX} + F_{XYY} + G_{XXY} + G_{YYY} + i(G_{XXX} + G_{XYY} - F_{XXY} - F_{YYY})\right]|_{(0,0)}.$$

According to [29], we now come to the following conclusion as a result of the analysis derived above.

Theorem 3.1. *Let the parameters $(\theta, \theta_1, \delta, k, h, r, a, q, \alpha, \rho) \in S$ and Δ_0 and ρ_0 be defined as in (3.1). L is defined in (3.5). If the parameter ρ varies in a vicinity of ρ_0 (correspondingly, Δ varies around Δ_0) and $L \neq 0$, then system (1.8) undergoes a Neimark–Sacker bifurcation at the fixed point E_2. Moreover, if $L < (>)0$, a stable (an unstable) smooth closed invariant curve can be bifurcated out and the bifurcation is supercritical (subcritical).*

3.2. Period-Doubling Bifurcation at the Fixed Point E_2

From Case 3 in the proof of Theorem 2.3 concerning the stability of the positive fixed point E_2, it can be observed that the dimensions of the stable and unstable manifolds of system (1.8) at E_2 undergo changes when Δ varies in the vicinity of Δ_1 (correspondingly, ρ varies in the vicinity of ρ_0) for $A > 4\Omega$, where

$$\Delta_1 = \frac{A_2 + \sqrt{A_2^2 - 4A_1}}{A_1}, \quad \rho_0 = (\Gamma(\alpha+1)\Delta_1)^{\frac{1}{\alpha}} \qquad (3.6)$$

Hence, a bifurcation may take place. Noting that $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ for $\Delta = \Delta_1$, it can be concluded that this bifurcation is of the period-doubling type. Let

$$S_1 = \{\theta, \theta_1, \delta, k, h, r, a, q, \alpha, \rho \in R_+^{10} | a < \frac{k\theta_1}{k+\delta}, A_2 < -2\sqrt{A_1}\}.$$

To analyze the period-doubling bifurcation of system (1.8) at the fixed point E_2, it suffices for us to consider Δ to vary in the neighbourhood of $\Delta_1 = \frac{A_2+\sqrt{A_2^2-4A_1}}{A_1}$. The proof for the case where $\Delta_2 = \frac{A_2-\sqrt{A_2^2-4A_1}}{A_1}$ is entirely analogous and will be omitted here. We now proceed as follows.

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Put $x_n = u_n - x^*$ and $y_n = v_n - y^*$, which transforms the fixed point E_2 to the origin $O(0,0)$. Consider ρ^* as a small perturbation of ρ, namely, $\rho^* = \rho - \rho_0$, with $0 < |\rho^*| \ll 1$. The perturbation takes system (1.8) into

$$\begin{cases} x_{n+1} = x_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(r(x_n + x^*) \left(1 - \frac{x_n+x^*}{k}\right) - \frac{\theta(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} \right), \\ y_{n+1} = y_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(\frac{\theta_1(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} - a(y_n + y^*) \right). \end{cases} \tag{3.7}$$

Set $\rho^*_{n+1} = \rho^*_n = \rho^*$, then system (4.7) can be seen as

$$\begin{cases} x_{n+1} = x_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(r(x_n + x^*) \left(1 - \frac{x_n+x^*}{k}\right) - \frac{\theta(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} \right), \\ y_{n+1} = y_n + \frac{(\rho^*+\rho_0)^\alpha}{\Gamma(\alpha+1)} \left(\frac{\theta_1(x_n+x^*)(y_n+y^*)}{\delta+x_n+x^*} - a(y_n + y^*) \right), \\ \rho^*_{n+1} = \rho^*_n. \end{cases} \tag{3.8}$$

Taylor expanding system (4.8) at $(x_n, v_n, \rho^*_n) = (0,0,0)$ results in

$$\begin{pmatrix} x_n \\ y_n \\ \rho^*_n \end{pmatrix} \to \begin{pmatrix} \epsilon_{100} & \epsilon_{010} & 0 \\ \zeta_{100} & \zeta_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ \rho^*_n \end{pmatrix} + \begin{pmatrix} M(x_n, y_n, \rho^*_n) + o(\rho_3^3) \\ N(x_n, y_n, \rho^*_n) + o(\rho_3^3) \\ 0 \end{pmatrix}, \tag{3.9}$$

where $\rho_3 = \sqrt{x_n^2 + y_n^2 + \rho_n^{*2}}$,

$$\begin{aligned} M(x_n, y_n, \rho^*_n) =& \epsilon_{200}x_n^2 + \epsilon_{020}y_n^2 + \epsilon_{002}\rho_n^{*2} + \epsilon_{110}x_n y_n + \epsilon_{101}x_n\rho^*_n + \epsilon_{011}y_n\rho^*_n \\ & + \epsilon_{300}x_n^3 + \epsilon_{030}y_n^3 + \epsilon_{003}\rho_n^{*3} + \epsilon_{210}x_n^2 y_n + \epsilon_{120}x_n y_n^2 \\ & + \epsilon_{201}x_n^2\rho^*_n + \epsilon_{102}x_n\rho_n^{*2} + \epsilon_{021}y_n^2\rho^*_n + \epsilon_{012}y_n\rho_n^{*2} + \epsilon_{111}x_n y_n\rho^*_n, \end{aligned}$$

$$\begin{aligned} N(x_n, y_n, \rho^*_n) =& \zeta_{200}x_n^2 + \zeta_{020}y_n^2 + \zeta_{002}\rho_n^{*2} + \zeta_{110}x_n y_n + \zeta_{101}x_n\rho^*_n + \zeta_{011}y_n\rho^*_n \\ & + \zeta_{300}x_n^3 + \zeta_{030}y_n^3 + \zeta_{003}\rho_n^{*3} + \zeta_{210}x_n^2 y_n + \zeta_{120}x_n y_n^2 \\ & + \zeta_{201}x_n^2\rho^*_n + \zeta_{102}x_n\rho_n^{*2} + \zeta_{021}y_n^2\rho^*_n + \zeta_{012}y_n\rho_n^{*2} + \zeta_{111}x_n y_n\rho^*_n, \end{aligned}$$

$$
\begin{aligned}
\epsilon_{100} &= 1 - \frac{\omega_1\omega_2}{\Gamma(\alpha+1)}, \epsilon_{010} = -\frac{\theta\omega_1 x^*}{\omega_8}, \\
\epsilon_{200} &= -\frac{\omega_1\omega_5}{\Gamma(\alpha+1)}, \epsilon_{020} = 0, \epsilon_{002} = \frac{\omega_1\omega_3\omega_4}{\Gamma(\alpha+1)}, \\
\epsilon_{110} &= \frac{\theta\omega_1\omega_7}{\Gamma(\alpha+1)}, \epsilon_{101} = -\frac{\alpha\omega_1\omega_2}{\omega_6}, \\
\epsilon_{011} &= -\frac{\alpha\theta x^*\omega_1}{\rho_0(\delta+x^*)\Gamma(\alpha+1)}, \\
\epsilon_{300} &= -\frac{\theta\omega_1}{\Gamma(\alpha+1)}\left(\frac{y^*}{(\delta+x^*)^3} - \frac{x^*y^*}{(\delta+x^*)^4}\right), \\
\epsilon_{030} &= 0, \epsilon_{003} = \frac{\omega_1\omega_3}{\Gamma(\alpha+1)}\left(\frac{\alpha^2}{2\rho_0^3} - \frac{\alpha}{3\rho_0^3} - \frac{\alpha^3}{6\rho_0^3}\right), \\
\epsilon_{210} &= -\frac{\theta\omega_1}{\Gamma(\alpha+1)}\left(\frac{x^*}{(\delta+x^*)^3} - \frac{1}{(\delta+x^*)^2}\right), \\
\epsilon_{120} &= 0, \epsilon_{021} = 0, \epsilon_{012} = \frac{\theta x^*\omega_1\omega_4}{\omega_8}, \\
\epsilon_{201} &= \frac{\alpha\omega_1\omega_5}{\omega_6}, \epsilon_{111} = \frac{\alpha\theta\omega_1\omega_7}{\omega_6}, \epsilon_{102} = \frac{\omega_1\omega_2\omega_4}{\Gamma(\alpha+1)}; \\
\zeta_{100} &= \frac{\theta_1\Omega_1\Omega_3}{\Gamma(\alpha+1)}, \zeta_{010} = 1 - \frac{\Omega_1\Omega_2}{\Gamma(\alpha+1)}, \\
\zeta_{200} &= -\frac{\theta_1\Omega_1\Omega_6}{\Gamma(\alpha+1)}, \zeta_{020} = 0, \zeta_{002} = \frac{\Omega_1\Omega_2\Omega_4}{\Gamma(\alpha+1)}, \\
\zeta_{110} &= \frac{\theta_1\Omega_1\Omega_8}{\Gamma(\alpha+1)}, \zeta_{101} = \frac{\alpha\theta_1\Omega_1\Omega_3}{\Omega_7}, \\
\zeta_{011} &= -\frac{\alpha\Omega_1\Omega_5}{\Omega_7}, \zeta_{300} = \frac{\theta_1\Omega_1}{\Gamma(\alpha+1)}\left(\frac{y^*}{(\delta+x^*)^3} - \frac{x^*y^*}{(\delta+x^*)^4}\right), \\
\zeta_{030} &= 0, \zeta_{003} = \frac{\Omega_1\Omega_4}{\Gamma(\alpha+1)}\left(\frac{\alpha^2}{2\rho_0^3} - \frac{\alpha}{3\rho_0^3} - \frac{\alpha^3}{6\rho_0^3}\right), \\
\zeta_{210} &= \frac{\theta_1\Omega_1}{\Gamma(\alpha+1)}\left(\frac{x^*}{(\delta+x^*)^3} - \frac{1}{(\delta+x^*)^2}\right), \\
\zeta_{120} &= 0, \zeta_{021} = 0, \zeta_{201} = -\frac{\alpha\theta_1\Omega_1\Omega_6}{\Omega_7}, \\
\zeta_{012} &= \frac{\Omega_1\Omega_2\Omega_5}{\Gamma(\alpha+1)}, \zeta_{102} = -\frac{\theta_1\Omega_1\Omega_2\Omega_3}{\Gamma(\alpha+1)}, \zeta_{111} = -\frac{\alpha\theta_1\Omega_1\Omega_8}{\Omega_7},
\end{aligned}
$$

$$\omega_1 = e^{\alpha \log \rho^*}, \omega_2 = \theta \left(\frac{y^*}{\delta + x^*} - \frac{x^* y^*}{(\delta + x^*)^2} \right) + r \left(\frac{x^*}{k} - 1 \right) + \frac{r x^*}{k},$$
$$\omega_3 = r x^* \left(\frac{x^*}{k} - 1 \right) + \frac{\theta x^* y^*}{\delta + x^*}, \omega_4 = \frac{\alpha}{2(\rho^*)^2} - \frac{\alpha^2}{2(\rho^*)^2},$$
$$\omega_5 = \theta \left(\frac{y^*}{(\delta + x^*)^2} - \frac{x^* y^*}{(\delta + x^*)^3} \right) - \frac{r}{k}, \omega_6 = \rho^* \Gamma(\alpha + 1),$$
$$\omega_7 = \frac{x^*}{(\delta + x^*)^2} - \frac{1}{\delta + x^*}, \omega_8 = \Gamma(\alpha + 1)(\delta + x^*),$$
$$\Omega_1 = e^{\alpha \log \rho^*}, \Omega_2 = \frac{\alpha}{2(\rho^*)^2} - \frac{\alpha^2}{2(\rho^*)^2},$$
$$\Omega_3 = \frac{y^*}{\delta + x^*} - \frac{x^* y^*}{(\delta + x^*)^2}, \Omega_4 = a y^* - \frac{\theta_1 x^* y^*}{\delta + x^*},$$
$$\Omega_5 = a - \frac{\theta_1 x^*}{\delta + x^*}, \Omega_6 = \frac{y^*}{(\delta + x^*)^2} - \frac{x^* y^*}{(\delta + x^*)^3},$$
$$\Omega_7 = \rho^* \Gamma(\alpha + 1), \Omega_8 = \frac{x^*}{(\delta + x^*)^2} - \frac{1}{\delta + x^*}.$$

Take

$$T = \begin{pmatrix} \epsilon_{100} & \epsilon_{010} \\ -1 - \epsilon_{100} & \lambda_2 - \epsilon_{100} \end{pmatrix},$$

which is invertible. Now, using the transformation

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = T \begin{pmatrix} \tilde{u_n} \\ \tilde{v_n} \end{pmatrix},$$

system (4.9) becomes

$$\begin{cases} \tilde{u}_{n+1} = -\tilde{u}_n + M(u_n - x^*, v_n - y^*, \rho_n^*), \\ \tilde{v}_{n+1} = \lambda_2 \tilde{v}_n + N(u_n - x^*, v_n - y^*, \rho_n^*). \end{cases} \tag{3.10}$$

System (4.10) has a center manifold $W^c(0,0,0)$ at $(0,0)$ in the neighborhood of $\rho^* = 0$, which can be deduced using the center manifold theorem and essentially expressed as follows:

$$W^c(0,0,0) = \left\{ (\tilde{u}_n, \tilde{v}_n, \rho^*) \in R^3 : \tilde{v}_n = \eta_1 \tilde{u}_n^2 + \eta_2 \tilde{u}_n \rho^* + o\left((|\tilde{u}_n| + |\rho^*|)^2 \right) \right\},$$

where

$$\eta_1 = \frac{\epsilon_{010} \left((1 + \epsilon_{100}) \epsilon_{200} + \epsilon_{010} \zeta_{200} \right) + \zeta_{020} (1 + \epsilon_{100})^2 - (1 + \epsilon_{100})(\epsilon_{110}(1 + \epsilon_{100}) + \epsilon_{010} \zeta_{110})}{1 - \lambda_2^2},$$

$$\eta_2 = \frac{(1 + \epsilon_{100})(\epsilon_{011}(1 + \epsilon_{100}) + \epsilon_{010} \zeta_{011})}{\epsilon_{010}(1 + \lambda_2)^2} - \frac{(1 + \epsilon_{100})(\epsilon_{101} + \epsilon_{010} \zeta_{101})}{(1 + \lambda_2)^2}.$$

So, system (4.10) restrained on the center manifold $W^c(0,0,0)$ has the following form:

$$\tilde{u}_{n+1} = -\tilde{u}_n + \theta_1 \tilde{u}_n^2 + \theta_2 \tilde{u}_n \rho^* + \theta_3 \tilde{u}_n^2 \rho^* + \theta_4 \tilde{u}_n \rho^{*2} + \theta_5 \tilde{u}_n^3 + o\left((|\tilde{u}_n| + |\rho^*|)^3 \right) =: Z(\tilde{u}_n, \rho^*),$$

where

$$
\begin{aligned}
\theta_1 &= \frac{\eta_2((\lambda_2-\eta_1)\epsilon_{200}-\eta_2\zeta_{200})}{1+\lambda_2} - \frac{\zeta_{020}(1+\eta_1)^2}{1+\lambda_2} - \frac{(1+\eta_1)((\lambda_2-\eta_1)\epsilon_{110}-\eta_2\zeta_{110})}{1+\lambda_2},\\
\theta_2 &= \frac{(\lambda_2-\eta_1)\epsilon_{101}-\eta_2\zeta_{101}}{1+\lambda_2} - \frac{(1+\eta_1)((\lambda_2-\eta_1)\epsilon_{011}-\eta_2\zeta_{011})}{\eta_2(1+\lambda_2)},\\
\theta_3 &= \frac{(\lambda_2-\epsilon_{100})\eta_1\epsilon_{101}-\epsilon_{010}\zeta_{101}}{1+\lambda_2} + \frac{((\lambda_2-\epsilon_{100})\epsilon_{011}-\eta_2\zeta_{011})(\lambda_2-\epsilon_{100})\eta_1}{\epsilon_{010}(1+\lambda_2)}\\
&\quad - \frac{(1-\epsilon_{100})((\lambda_2-\epsilon_{100})\epsilon_{111}-\epsilon_{010}\zeta_{111})}{1+\lambda_2} + \frac{\epsilon_{010}((\lambda_2-\epsilon_{100})\epsilon_{201}-\epsilon_{010}\zeta_{201})}{1+\lambda_2}\\
&\quad - \frac{\zeta_{021}(1-\epsilon_{100})^2}{1+\lambda_2} + \frac{2\epsilon_{100}\eta_2((\lambda_2-\epsilon_{100})\epsilon_{200}-\epsilon_{010}\zeta_{200})}{1+\lambda_2}\\
&\quad + \frac{2\zeta_{020}\eta_2(1+\epsilon_{100})(\lambda_2-\epsilon_{100})}{1+\lambda_2} + \frac{\eta_2((\lambda_2-\epsilon_{100})\epsilon_{110}-\epsilon_{010}\zeta_{110})(\lambda_2-1-2\epsilon_{100})}{1+\lambda_2},\\
\theta_4 &= \frac{\eta_2((\lambda_2-\epsilon_{100})\epsilon_{101}-\epsilon_{010}\zeta_{101})}{1+\lambda_2} + \frac{((\lambda_2-\epsilon_{100})\epsilon_{011}-\eta_2\zeta_{011})(\lambda_2-\epsilon_{100})\eta_2}{\epsilon_{010}(1+\lambda_2)}\\
&\quad + \frac{2\epsilon_{010}\eta_2((\lambda_2-\epsilon_{100})\epsilon_{200}-\epsilon_{010}\zeta_{200})}{1+\lambda_2} + \frac{2\zeta_{020}\eta_2(1+\epsilon_{100})(\lambda_2-\epsilon_{100})}{1+\lambda_2}\\
&\quad + \frac{\eta_2(\lambda_2-1-2\epsilon_{100})((\lambda_2-\epsilon_{100})\epsilon_{110}-\epsilon_{010}\zeta_{110})}{1+\lambda_2},\\
\theta_5 &= \frac{2\epsilon_{010}\eta_1((\lambda_2-\epsilon_{100})\epsilon_{200}-\epsilon_{010}\zeta_{200})}{1+\lambda_2} + \frac{\eta_1(\lambda_2-1-2\epsilon_{100})((\lambda_2-\epsilon_{100})\epsilon_{200}-\epsilon_{010}\zeta_{200})}{2+\lambda_2}\\
&\quad + \frac{2\zeta_{020}\eta_1(\lambda_2-\epsilon_{100})(1+\epsilon_{100})}{1+\lambda_2} + \frac{\eta_2^2((\lambda_2-\epsilon_{100})\epsilon_{300}-\epsilon_{010}\zeta_{300})}{1+\lambda_2}\\
&\quad - \frac{\eta_2(1+\epsilon_{100})((\lambda_2-\epsilon_{100})\epsilon_{210}-\epsilon_{010}\zeta_{210})}{1+\lambda_2}.
\end{aligned}
$$

In order for the period-doubling bifurcation to occur, the two determinating quantities ζ_1 and ζ_2 must be nonzero, where

$$
\zeta_1 = \left(\frac{\partial^2 Z}{\partial\tilde{u}\partial\rho^*} + \frac{1}{2}\frac{\partial Z}{\partial\rho^*}\frac{\partial^2 Z}{\partial\tilde{u}^2}\right)\Bigg|_{(0,0)}, \quad \zeta_2 = \left(\frac{1}{6}\frac{\partial^3 Z}{\partial\tilde{u}^3} + \left(\frac{1}{2}\frac{\partial^2 Z}{\partial\tilde{u}^2}\right)^2\right)\Bigg|_{(0,0)}.
$$

Finally, the outcome of the above analysis may be summarized as follows.

Theorem 3.2. *Assume that the parameters* $(\theta,\theta_1,\delta,k,h,r,a,q,\alpha,\rho)\in S_1$, *with* Δ_0 *and* ρ_0 *defined as in (3.6). When the parameter* ρ *varies within a neighborhood of* ρ_0 *(and correspondingly,* Δ *changes around* Δ_0*) and* $\zeta_1\zeta_2\neq 0$, *system (1.8) undergoes a period-doubling bifurcation at the fixed point* E_2. *Moreover, if* $\zeta_2>0$ $(\zeta_2<0)$, *the period-two orbit bifurcating from* E_2 *is stable (unstable).*

4. Numerical Simulation

In this section, we perform numerical simulations for the dynamical behavior of systems (1.8) using Matlab, aiming to provide readers with a more intuitive understanding to the dynamics of systems (1.8).

4 NUMERICAL SIMULATION

First, we fix the parameter values as $\alpha = 0.6$, $a = 0.2$, $\theta = 1.5$, $\theta_1 = 1$, $\delta = 5$, $k = 3$, $r = 0.2$, and let $\rho \in (0, 10)$. Figure 1 presents the bifurcation diagram of system (1.8) initiated from the point $(u_0, v_0) = (0.6, 0.2)$, from which it is evident that system (1.8) undergoes a Neimark–Sacker bifurcation at the critical value $\rho_0 = 1.591936$.

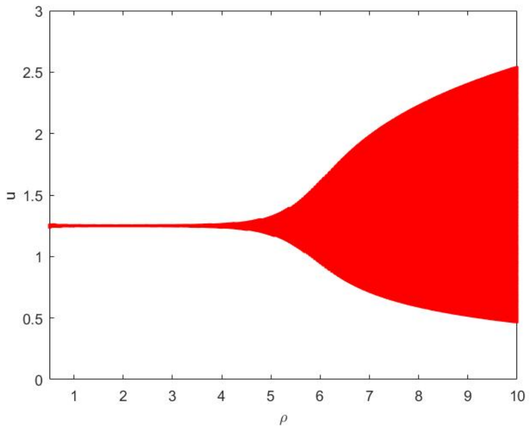


Figure 1: The existence of a Neimark–Sacker bifurcation in system (1.8) as ρ takes values from 0 to 10.

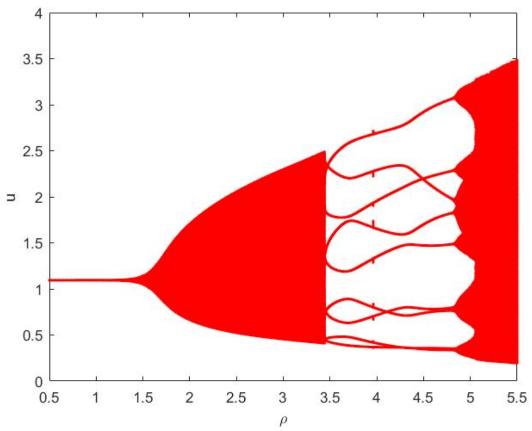
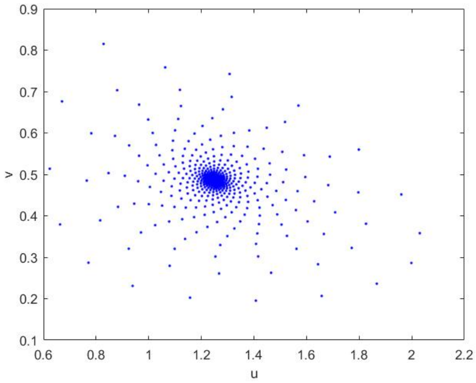


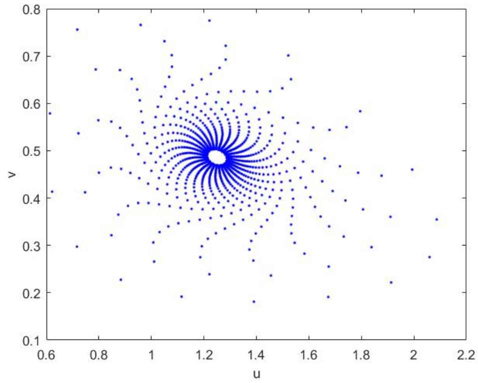
Figure 2: The existence of a period-doubling bifurcation in system (1.8) as ρ takes values from 0.5 to 5.5.

With the values $\alpha = 0.58; a = 0.304; \theta = 2.2; \theta_1 = 1.415; \delta = 4; k = 3; r = 0.865$, Figure 2 is the bifurcation diagram of system (1.8) starting from the point $(u_0, v_0) = (0.7, 0.2)$. We can clearly observe that system (1.8) undergoes a period-doubling bifurcation at the critical value $\rho_0 = 0.81683$.

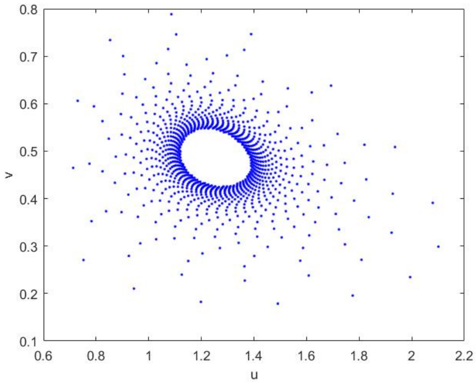
Figure 3 depicts the phase diagram of system (1.8) starting from the point $(u_0, v_0) = (2.551020, 0.287415)$ with parameters $\alpha = 0.6, c = 0.4, d = 0.5, e = 0.6, k = 6, m = 0.3, r = 0.2$. We can observe that as ρ increases, the equilibrium point gradually transits from stable state to unstable state, and an invariant closed curve emerges.



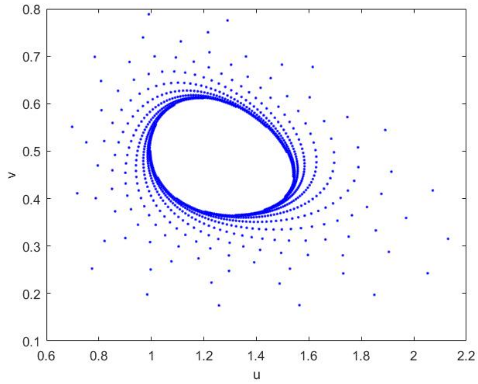
(a) ρ= 5



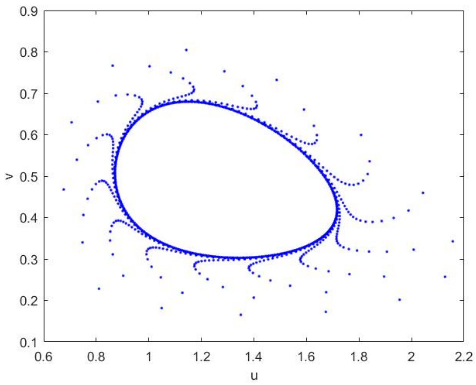
(b) ρ= 5.6



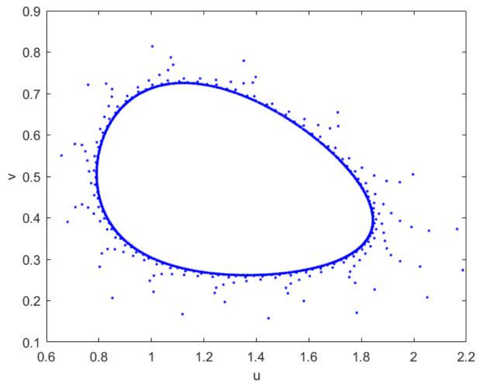
(c) ρ= 5.8



(d) ρ= 6.0



(e) ρ= 6.3



(f) ρ= 6.6

Figure 3: Phase portraits of system (1.6) with $\alpha = 0.6, a = 0.2, \theta = 1.5, \theta_1 = 1, \delta = 5, k = 3, r = 0.2$ and different ρ when the initial value $(x_0, y_0) = (2.551020, 0.287415)$.

5. Conclusion

In this paper, we explore the discrete form of a fractional-order predator-prey model incorporating the Holling type II functional response. Following the simplification of system (1.2), we have enhanced the model's realism. Given the current scarcity of effective approaches for investigating the dynamics of fractional-order differential systems, we adopt the piecewise constant approximation method to discretize the continuous fractional-order predator-prey system (1.5) in this study. We then analyze its dynamical characteristics and discuss the types of bifurcations exhibited by the system. Given parameter conditions, we completely formulate the existence and stability of nonnegative fixed points $E_0 = (0, 0)$, $E_1 = (k, 0)$ and $E_2 = (\frac{a\delta}{\theta_1 - a}, \frac{r\delta\theta_1}{\theta(\theta_1 - a)}\left(1 - \frac{a\delta}{k(\theta_1 - a)}\right))$ for $a < \frac{k\theta_1}{k+\delta}$.

We derive sufficient conditions for the occurrence of Neimark-Sacker and period-doubling bifurcations at the fixed point E_2 within certain parameter spaces. Additionally, we analyze the stability and direction of the resulting closed orbits. Finally, numerical simulations are presented to illustrate intriguing dynamical behaviors associated with these bifurcations.

Through analytical and numerical investigations of the positive fixed point E_2, we arrive at the following conclusions: When the parameter ρ exceeds a critical threshold, the system transitions to a stable limit cycle. This indicates that under appropriate conditions, both predator and prey populations are capable of coexisting in a dynamically stable state.

Funding: This work was partially supported by the (Grant No. 61473340), the Distinguished Professor Foundation of Qianjiang Scholars in Zhejiang Province (Grant No. F708108P01), and the Natural Science Foundation of Zhejiang University of Science and Technology (Grant No. 0401108P10)..

Data Availability Statement: There are no applicable data associated with this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Authors' Contributions: All authors contributed equally and significantly in writing this paper. All authors have read and approved the final manuscripNational Natural Science Foundation of Chinat.

Use of AI Tools Declaration: The authors declare that no artificial intelligence (AI) tools were utilized in the creation of this article.

Apendix

In this section, we primarily introduce the definition and some conclusions of Caputo fractional derivative that are necessary for our subsequent research.

5 CONCLUSION

Definition 5.1. *[30] Under the definition of Caputo fractional derivative, the fractional derivative of function* $f(\xi) \in AC^n([0, +\infty], \mathbb{R})$ *is given as*

$$ {}_0^C D_\xi^\alpha f(\xi) = \int_0^\xi \frac{f^{(n)}(\vartheta)}{\Gamma(n-\alpha)(\xi-\vartheta)^{\alpha-n+1}} d\vartheta, $$

where α *denotes the order of the fractional derivative, and* n *is the integer closest to* α *such that* $n < \alpha \leq n+1$.

When $n = 1$, *the fractional derivative* ${}_0^C D_\xi^\alpha f(\xi)$ *takes the form of*

$$ {}_0^C D_\xi^\alpha f(\xi) = \int_0^\xi \frac{f'(\vartheta)}{\Gamma(1-\alpha)(\xi-\vartheta)^{\alpha}} d\vartheta. $$

Definition 5.2. *[29] The Mittag–Leffler function, denoted as Mi, where the order i of Mi is positive, is defined as follows:*

$$ M_i(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta_j}{\Gamma(ji+1)}, \zeta_j \in \mathbb{C} $$

when the series converges.

Definition 5.3. *[31] Let* $Q(u, v)$ *be a fixed piont of system (1.8) with multipliers* λ_1 *and* λ_2.

(i) If $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$, *then the fixed point* $Q(u, v)$ *is called a sink, and a sink is locally asymptotically stable.*

(ii) If $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$, *the the fixed point* $Q(u, v)$ *is called a source, and a source is locally unstable.*

(iii) If $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*), then the fixed point* $Q(u, v)$ *is called a saddle.*

(iv) If either $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$, *then the fixed point* $Q(u, v)$ *is called non-hyperbolic.*

Lemma 1. [29] Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then, the following statements hold.

(i) If $F(1) > 0$, then

(i.1)$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;

(i.2) $\lambda_1 = -1$ and $\lambda_2 \neq -1$ if and only if $F(-1) = 0$ and $B \neq 2$;

(i.3) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $F(-1) < 0$;

(i.4) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;

(i.5) λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$ if and only if $-2 < B < 2$ and $C = 1$;

(i.6) $\lambda_1 = \lambda_2 = -1$ if and only if $F(-1) = 0$ and $B = 2$.

(ii) If $F(1) = 0$, namely, 1 is a root of $F(\lambda) = 0$, then the another root λ satisfies $|\lambda| = (<, >)1$ if and only if $|C| = (<, >)1$.

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(iii) If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in $(1, \infty)$. Moreover,

(iii.1) the other root λ satisfies $\lambda < (=) - 1$ if and only if $F(-1) < (=)0$;

(iii.2) the other root $-1 < \lambda < 1$ if and only if $F(-1) > 0$.

CRediT

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