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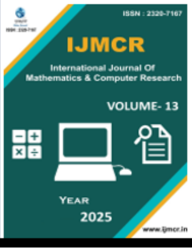

QUADRATIC TYPE GENERALOZED Z-CONTRACTION WITH RESPECT TO A SIMULATION FUNCTION AND COMMON FIXED POINT THEOREM

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Abstract. In this paper, we introduce quadratic type generalized Z-contraction with respect to a simulation function and study the existence of common fixed points of such mappings in complete b- metric spaces. We extend it to a sequence of self maps. We infer some corollaries from our main result and provide examples to verify our results.

MSC (2020): 47H10; 54H25.

Keywords: Quadratic type generalized Z-contraction pair of self maps; simulation function; b-metric space; fixed point; common fixed point.

1. Introduction

The Banach contraction principle is a fundamental results in fixed point theory. Due to its importance, various mathematician studied many interesting extensions and generalizations, (see [7, 13, 12, 17]). It is considered to be the connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all Engineering areas, b-metric spaces were introduced by Bakhtin [3] and Czerwik [9]. In 2015, [14] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theorems via simulation functions.

2. Preliminaries

We present some definitions which will be useful in the sequel.

Definition 2.1. [9] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be b-metric space if the following conditions are satisfied:

(1) $0 \leq d(x, y)\ \forall\ x, y \in X$ *and* $d(x, y) = 0 \iff x = y$,

(2) $d(x, y) = d(y, x)\ \forall\ x, y \in X$, and

(3) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]\ \forall\ x, y, z \in X$.

In this case, the pair (X, d) is called a b-metric space with coefficient s. Here, we observe that every metric space is a b-metric space, with $s = 1$

Definition 2.2. [5] Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X.

(1) A sequence x $_{n}\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0 \; as \; n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(2) A Sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0 \; as \; n, m \to \infty$.

(3) A b-metric space (X, d) is said to be a complete b- metric space if every b-Cauchy sequence in X is b-convergent.

(4) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X \; then \; z \in B$.

Definition 2.3. [14] Let (X, d) be a metric space and $f : X \to X$ be a selfmap of X. We say that f is a Z-contraction with respect to ξ, if there exists a simulation function ξ such that

$$\xi(d(fx, fy), d(x, y)) \geq 0 \; \forall \; x, y \in X$$

Definition 2.4. [14] Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be two self mappings. If there exists a simulation function ξ such that

$$\xi(s^4 d(fx, gy), M(x, y)) \geq 0,$$

where

$$M(x, y) = max\left\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2s}\left[d(x, gy) + d(y, fx)\right]\right\}, \forall \; x, y \in X.$$

then we say that (f, g) is generalized Z-contraction pair of maps. He proved common fixed point theorem for above contraction

Definition 2.5. [5] (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is b-continuous at a point $x \in X$, if it is b-sequentially continuous at x i.e., whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.

Definition 2.6. [14] A simulation function is a mapping
$\xi : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$ satisfying the following conditions:
(ξ_1) $\xi(0, 0) = 0$;
(ξ_2) $\xi(t, s) < s - t \; \forall \; s, t > 0$;
(ξ_3) if $\{t_n\}$,$\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, \infty) \; then \; \limsup_{n\to\infty} \xi(t_n, s_n) > 0.$$

Motivated by the works of Olgun,Bicer and Alyildiz [15], [2] we introduce a generalized Z-contraction pair of maps with respect to simulation function and established common fixed point theorem.

3. Main Result

Definition 3.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be two self mappings. If there exists a simulation function ξ such that

$$\xi(s^4 d(fx, gy), M(x, y)) \geq 0,$$

where

$$M(x,y) = max\left\{d^2(x,y), d(x,fx).d(y,gy), \frac{1}{2s}\left[d^2(x,gy)+d^2(y,fx)\right]\right\}, \forall\, x,y \in X,$$

then we say that (f,g) is generalized quadratic type Z-contraction. If $g=f$ and $s=2$, then we say that f are quadratic type generalized Z-contraction map of a metric space X.

Example: Let $X=[0,1]$ and $d\colon X\times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x=y \\ x+y & \text{if } x\neq y \end{cases}$$

Clearly (X,d) is a b-metric space with coefficient $S=2$. We define $f,g:X\to X$ by $fx=\frac{x}{10}$ for all $x\in[0,1]$ and

$$gx = \begin{cases} 0 & \text{if } x\in[0,1]/\{\frac{1}{2}\} \\ \frac{1}{20} & \text{if } x=\frac{1}{2}. \end{cases}$$

Now, we define $\xi:[0,\infty)\times[0,\infty)\to[0,\infty)$ by $\xi(t,s)=\lambda s-t$ where $\lambda\ >1$. We have the following possible cases.

Case(I): $x=\frac{1}{2}$, $y=\frac{1}{2}$.

In this case, $fx=\frac{1}{20}$, $gy=\frac{1}{20}$, $s^4d(fx,gy)=0$ and

$$\begin{aligned} M(x,y) &= Max\left\{(x+y)^2, (x+\frac{x}{10}).(y+\frac{1}{20}), \frac{1}{2s}[(x+\frac{1}{20})^2+(y+\frac{x}{10})^2]\right\} \\ &= max\left\{\left(\frac{1}{2}+\frac{1}{2}\right)^2, \left(\frac{1}{2}+\frac{1}{20}\right)^2, \frac{1}{2}\left[\left(\frac{11}{20}\right)^2\right]\right\} \\ &= max\left\{1, \left(\frac{11}{20}\right)^2\right\} \\ &= 1. \end{aligned}$$

$$\begin{aligned} \xi(t,s) &= \lambda s - t \\ \xi(t,s) &= \lambda M(x,y) - s^4 d(fx,gy) \\ &= 4-0 \\ &= 4 \\ &> 0. \end{aligned}$$

Case(II): $x \neq \frac{1}{2}$, $y = \frac{1}{2}$, $fx = \frac{x}{10}$, $gy = \begin{cases} 0 \text{ if } x \in [0,1]/\{\frac{1}{2}\} \\ \frac{1}{20} \text{ if } x = \frac{1}{2}. \end{cases}$

$$\begin{aligned} s^4 d(fx, gy) &= 2^4 (\frac{x}{10} + \frac{1}{20}) \\ &= \left\{ \frac{16}{20}(2x+1) \right\} \\ &= \frac{48}{20} \\ &= 2.4 \end{aligned}$$

$$\begin{aligned} M(x,y) &= max \left\{ d^2(x,y), d(x,fx).d(y,gy), \frac{1}{2s}[d^2(x,gy) + d^2(y,fx)] \right\} \\ &= max \ \left\{ \left(x + \frac{1}{2}\right)^2, \frac{11x}{10}.\frac{11}{20}, \frac{1}{4}\left[\left(\frac{20x+1}{20}\right)^2 + \left(\frac{5+x}{10}\right)^2\right] \right\} \\ &= max \ \left\{ (x+y)^2, \left(x + \frac{x}{10}\right) \cdot \left(y + \frac{1}{20}\right), \frac{1}{4}\left[\left(x + \frac{1}{20}\right)^2 + \left(\frac{1}{2} + \frac{x}{10}\right)^2\right] \right\} \\ &= max \left\{ \left(\frac{3}{2}\right)^2, \frac{121}{200}, \frac{1}{4}\left[\left(\frac{21}{20}\right)^2 + \left(\frac{6}{10}\right)^2\right] \right\} \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} \xi(t,s) &= \lambda s - t \\ &= 4M(x,y) - 2.4 \\ &= 4 \cdot 2.5 - 2.4 \\ &= 6.6 \\ &> 0. \end{aligned}$$

Case(III): $x \neq \frac{1}{2}$, $y \neq \frac{1}{2}$.
In this case $fx = \frac{x}{10}$, $gy = 0$, $s^4 d(fx, gy) = 2^4. \left(\frac{x}{10} + \frac{16x}{10}\right)$

$$\begin{aligned} and M(x,y) &= max \left\{ (x+y)^2, \left(x + \frac{x}{10}\right) \cdot \left(y + \frac{1}{20}\right), \frac{1}{4}\left[\left(x + \frac{1}{20}\right)^2 + \left(y + \frac{x}{10}\right)^2\right] \right\} \\ &= max \left\{ 2^2 = 4, \left(\frac{11x}{10}\right) \cdot \left(\frac{20y+1}{20}\right), \frac{1}{4}\left[\left(\frac{20x+1}{20}\right)^2 + \left(\frac{10y+x}{10}\right)^2\right] \right\} \\ &= max \left\{ 4, \left(\frac{11}{10}\right) \cdot \left(\frac{21}{20}\right), \frac{1}{4}\left[\left(\frac{21}{20}\right)^2 + \left(\frac{11}{10}\right)^2\right] \right\} \\ &= 4. \end{aligned}$$

$$\begin{aligned}\xi(t,s) &= \lambda s - t\\ &= 4M(x,y) - 2.4\\ &= 4\cdot 4 - 1.6\\ &= 14.4\\ &> 0.\end{aligned}$$

Case(IV): $x = \frac{1}{2}$, $y \neq \frac{1}{2}$.
In this case $fx = \frac{x}{10}$, $gy = 0$, $s^4 d(fx, gy) = \frac{16}{20}$,

$$\begin{aligned}M(x,y) &= max\left\{(x+y)^2, (x+\frac{x}{10}).(y+\frac{1}{20}), \frac{1}{4}\left[\left(x+\frac{1}{20}\right)^2 + \left(y+\frac{1}{20}\right)^2\right]\right\}\\ &= max\left\{\left(y+\frac{1}{2}^2\right), \left(\frac{1}{2}+\frac{1}{20}\right)\cdot\left(y+\frac{1}{20}\right), \frac{1}{4}\left[\left(\frac{11}{20}\right)^2 + \left(\frac{21}{20}\right)^2\right]\right\}\\ &= max\ \left\{\left(\frac{3}{2}\right)^2, \frac{11.21}{400}, \frac{1}{4}\left[\left(\frac{11}{20}\right)^2 + \left(\frac{21}{20}\right)^2\right]\right\}\\ &= 2.5\end{aligned}$$

$$\begin{aligned}\xi(t,s) &= \lambda s - t\\ &= 4.25 - .8\\ &= 9.2\\ &> 0.\end{aligned}$$

Proposition 3.2. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be two self maps. Assume that f, g are generalized quadratic type Z-contraction. Then u is a fixed point f if and only if u is a fixed point of g. Moreover in that case u is unique.*

Proof. Let u be fixed point of f i.e. $fu = u$. Suppose $gu \neq u$. We consider

$$\xi(s^4 d(fu, gu), M(u, fu)) = \xi(s^4 d(u, gu), M(u, u) \geq 0, \tag{3.1}$$

where

$$\begin{aligned}M(u,u) &= max\left\{d^2(u,u), d(u,fu).d(u,gu), \frac{1}{2s}[d^2(u,gu) + d^2(u,fu)]\right\}\\ &= max\left\{0, 0, \frac{1}{2s}d^2(u,gu)\right\}\\ &= \left\{\frac{1}{2s}d^2(u,gu)\right\}.\end{aligned}$$

Now using the value of $M(u,u)$ in 3.1, we get

$$0 \leq \xi(s^4 d(u,gu), M(u,u)) = \xi(s^4 d(u,gu), d(u,gu)) < d(u,gu) - s^4 d(u,gu) \leq 0$$

a contradiction. Hence $gu = u$, so that u is common fixed point of f and g. Similarly, it is easy to see that if u is fixed point of g then u is fixed point of f

also. Suppose u and v are two common fixed point of f and g with $u \neq v$. From the inequality

$$\xi(s^4 d(fx, gy), M(x, y)) \geq 0$$

where

$$M(x, y) = max \left\{ d^2(x, y), d(x, fx).d(y, gy), \frac{1}{2s} \left[d^2(x, gy) + d^2(y, fx) \right] \right\}.$$

We have

$$\xi(s^4 d(u, v), M(u, v)) = \xi(s^4 d(fu, gv).M(u, v)) \geq 0, \tag{3.2}$$

where

$$\begin{aligned} M(u, v) &= max \left\{ d^2(u, v), d(u, fu).d(v, gv), \frac{1}{2s} \left[d^2(u, gv) + d^2(v, fu) \right] \right\} \\ &= max \left\{ d^2(u, v), 0, 0, \frac{1}{2s} \left[d^2(u, gv) + d^2(v, fv) \right] \right\} \\ &= \left\{ d^2(u, v) \right\}. \end{aligned}$$

Now using the value of $M(u, v)$ in 3.1, we get

$$\begin{aligned} 0 &\leq \xi(s^4 d(fu, gv), M(u, v)) \\ &= \xi(s^4 d(u, v), d^2(u, v) \\ &< d(u, v) - s^4 d(u, v) \leq 0, \end{aligned}$$

a contradiction. Hence $u = v$. Therefore the proposition follows. □

Now we state and prove the following quadratic type Z-contraction fixed point theorem.

Theorem 3.3. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $f, g : X \to X$ be two self maps. Assume that f, g are generalized quadratic type Z-contraction. Then f, g have a unique common fixed point in X, provided either f or g is b-continuous.*

Proof. Let $x_0 \in X$ be arbitrary. Since $fX \subseteq X$ and $gX \subseteq X$, there exists x_1, $x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$. Similarly there exist x_3, $x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$. In general, we construct a sequence x_n by $fx_{2n} = x_{2n+1}$, $gx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2...$
Suppose that $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = fx_{2n}$ so that x_{2n} is a fixed point of f. Hence by proposition 2.1, we have x_{2n} is a fixed point of g also so that x_{2n} is a common fixed point of f and g.
Similarly, if $x_{2n+1} = x_{2n+2}$ for some n. Then x_{2n+1} is a common fixed point of f and g. Hence in both the cases, f and g have common fixed point. Hence without loss of generality, we assume that $x_{2n} \neq x_{2n+1}$ for all n.

Now, we consider

$$\begin{aligned} \xi(s^4 d(x_{2n+1}, x_{2n+2}), M(x_{2n}, x_{2n+1})) &= \xi(s^4 d(fx_{2n}, gx_{2n+1}), M(x_{2n}, x_{2n+1})) \\ &\geq 0, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= max\{d^2(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}).(d(x_{2n+1}, gx_{2n+1}), \\ &\quad \frac{1}{2s}[d^2(x_{2n}, gx_{2n+1}) + (d^2 x_{2n+1}, fx_{2n})]\} \\ &= max\{d^2(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}).d(x_{2n+1}, x_{2n+2}), \\ &\quad \frac{1}{2s}[d^2(x_{2n}, x_{2n+2}) + d^2(x_{2n+1}, x_{2n+1}]\} \\ &= max\{d^2(x_{2n}, x_{2n+1}), d^2(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Suppose that for some $n \in \mathbb{N}$, $d^2(x_{2n}, x_{2n+1}) < d^2(x_{2n+1}, x_{2n+2})$ Hence, we have $M(x_{2n}, x_{2n+1}) = d^2(x_{2n+1}, x_{2n+2})$. Now, using the value of $M(x_{2n}, x_{2n+1})$ and from in eq. 3.3 we have

$$\begin{aligned} 0 &\leq \xi(s^4 d((x_{2n+1}, x_{2n+2}), M(x_{2n+1}, x_{2n+1})) \\ &= \xi(s^4 d(x_{2n+1}, x_{2n+2}), d^2 x_{2n+1}, x_{2n+2})) \\ &< d^2(x_{2n+1}, x_{2n+2}) - S^4 d(x_{2n+1}, x_{2n+2}) \\ &\leq 0. \end{aligned}$$

Therefore $d^2(x_{2n}, x_{2n+1}) \geq d^2(x_{2n+1}, x_{2n+2})$. Similarly, we can prove that

$$d^2(x_{2n+1}, x_{2n+2}) \geq d^2(x_{2n+2}, x_{2n+3}).$$

Hence

$$d^2(x_{n+1}, x_{n+2}) \leq d^2(x_n, x_{n+1}) \; for \; all \; n.$$

Therefore $\{d(x_n, x_{n+1})\}$ is a decreasing sequence and bounded below by zero . Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose that $r > 0$. Now using the condition ξ_3 with $t_n = d(x_{2n+1}, x_{2n+2})$ and $s_n = d(x_{2n}, x_{2n+1})$ we have $0 \leq \limsup\limits_{n\to\infty} \xi(s^4 d(x_{2n+1}, x_{2n+2}), M(x_{2n}, x_{2n+1})) < 0$, a contradiction. Therefore $r = 0$. So

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.4}$$

Now we prove that $\{x_n\}$ is a b-Cauchy sequence. For this it is sufficient to show that the subsequence $\{x_{2n}\}$ is a b-Cauchy sequence in X. Suppose that $\{x_{2n}\}$ is not a b-Cauchy sequence . Then there exists an $\epsilon > 0$ and the sequences positive integers $2m_k$ and $2n_k$ with $2n_k > 2m_k > k$ such that

$$d(x_{2mk}, x_{2nk}) \geq \epsilon \; and \; d(x_{2mk}, x_{2nk} - 2) < \epsilon. \tag{3.5}$$

Now, we consider

$$\begin{aligned} \xi(s^4 d(x_{2nk+1}, x_{2mk}), M(x_{2nk}, x_{2mk-1})) &= \xi(s^4 d(fx_{2nk+1}, gx_{2mk-1}), M(x_{2nk}, x_{2mk-1}) \\ &\geq 0, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} M(x_{2n_k}, x_{2m_{k-1}}) &= max\{d^2(x_{2n_k}, x_{2m_{k-1}}), d(x_{2n_k}, fx_{2n_k}).d(x_{2m_{k-1}}, gx_{2m_{k-1}}),\\ &\qquad \frac{1}{2s}[d^2((x_{2n_k}, gx_{2m_{k-1}}) + d^2(fx_{2n_k}, x_{2m_{k-1}})]\}\\ &= max\{d^2(x_{2n_k}, x_{2m_{k-1}}), d(x_{2n_k}, x_{2n_{k+1}}).d(x_{2m_k}, x_{2m_{k-1}}),\\ &\qquad \frac{1}{2s}[d^2(x_{2n_k}, x_{2m_k}) + d^2(x_{2n_{k+1}}, x_{2m_{k-1}})]\} \end{aligned}$$

Now, we consider the following two cases.

Case(I): $s = 1$.

In this case (X, d) is a metric space. Then there exists $\epsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that $2n_k > 2m_k \geq k$ with $d(x_{2m_k}, x_{2n_k}) \geq \epsilon$ and $d(x_{2m_k}, x_{2n_{k-2}}) < \epsilon$.

Hence we have

$$\begin{aligned} M(x_{2n_k}, x_{2m_{k-1}}) &= max\{d^2(x_{2n_k}, x_{2m_{k-1}}), d(x_{2n_k}, x_{2n_{k+1}}).d(x_{2m_{k-1}}, x_{2m_k}),\\ &\qquad \frac{d^2(x_{2n_k}, x_{2m_k}) + d^2(x_{2m_{k-1}}, x_{2n_{k+1}})}{2}\}. \end{aligned} \tag{3.7}$$

On taking limit as $k \to \infty$. we have $\lim_{k\to\infty} M(x_{2n_k}, x_{2m_{k-1}}) = \epsilon$. Using the condition ξ_3 with $t_n = d(x_{2n_{k+1}}, x_{2m_k})$ and $s_n = M(x_{2n_k}, x_{2m_{k-1}})$, we have
$0 \leq \limsup_{k\to\infty} \xi(d(x_{2n_{k+1}}, x_{2m_k}), M(x_{2n_k}, x_{2m_{k-1}})) < 0$, a contradiction.

Case(II): $s > 1$

In this case, by Lemma 1.2 [2] there exists $\epsilon > 0$ and sequences of positive integers $2n_k$ and $2m_k$ such that $2n_k > 2m_k \geq k$ with $d(2m_k, 2n_k) \geq \epsilon$ and $d(2m_k, 2n_{k-2}) < \epsilon$ satisfying (I) - (IV) of Lemma 1.2 [2].

Again taking $\lim_{k\to\infty}$ in the 3.7 and using conditions (I) -(IV) of Lemma 1.2 [2], we have

$$\lim_{k\to\infty} M(x_{2n_k}, x_{2m_{k-1}}) = max\left\{s^2\epsilon, 0, \frac{s\epsilon + s^3\epsilon}{2s}\right\} = s^2\epsilon.$$

$$Hence\ from\ (1.3), we\ have\ 0 \leq\ \xi(s^4 d(fx_{2n_k}, gx_{2m_{k-1}}), M(x_{2n_k}, x_{2m_{k-1}})).$$

Now we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \xi(s^4 d(fx_{2n_k}, gx_{2m_{k-1}}), M(x_{2n_k}, x_{2m_{k-1}}))\\ &\leq \limsup_{k\to\infty}\ [M(x_{2n_k}, x_{2m_{k-1}}) - s^4 d(x_{2n_{k+1}}, x_{2m_k})\ ]\\ &\leq \limsup_{k\to\infty} M(x_{2n_k}, x_{2m_{k-1}}) - s^4 \liminf_{k\to\infty} d(x_{2n_{k+1}}, x_{2m_k}) \leq s^2\epsilon - s^4\frac{\epsilon}{s}\\ &< 0, \end{aligned}$$

a contradiction. Therefore by case (I) and case (II), we have x_{2n} is a b-Cauchy sequence in (X, d). Hence $\{x_n\}$ is a b-Cauchy sequence in (X, d). Since X is a complete b-metric space, we have x_n is b-convergent to some point x (say) in X. Therefore $x = lim_{n\to\infty} x_{2n+1} = lim_{n\to\infty} fx_{2n}\ and\ x = lim_{n\to\infty} x_{2n+2} = lim_{n\to\infty} gx_{2n+1}\ so\ that\ lim_{n\to\infty} fx_{2n} = x = lim_{n\to\infty} gx_{2n+1}$.

We, assume that f is b-continuous .
Since $x_{2n} \to x \; as \; n \to \infty$, we have $fx_{2n} \to fx \; as \; n \to \infty$. Hence $0 \leq d(x, fx) \leq s(d(x, fx_{2n}) + d(fx_{2n}, fx)) \to 0 \; as \; n \to \infty$ so that $d(x, fx) = 0$. Hence x is a fixed point of f.
Now by proposition 2.1, we have x is a unique common fixed point of f and g.
Similarly, we can prove that x is a unique common fixed point of f and g is a b-continuous. □

Now we state and prove the following fixed point theorem.

Theorem 3.4. *Let (X, d) be a complete b-metric space with coefficients $s \geq 1$. Let $\{f_n\}$ be a sequence of self maps defined on a b-metric space (X, d). Assume that for each $i \neq 1, (f_1, f_i)$ is a generalized quadratic type Z-contraction . Then the sequence $\{f_i\}$ has a unique common fixed point in X, provided that at least one of the maps f_i is b-continuous.*

Proof. Fix $i \neq 1$. Since (f_1, f_i) is quadratic generalized Z-contraction pair of maps, by Theorem 2.1, we have (f_1, f_i) is a unique common fixed point in X.
Hence the conclusion of the theorem follows. □

Corollary 3.1. Let (X, d) be a complete b-metric space with coefficients $s \geq 1 \; and \; f : X \to X$ be a self maps on X. If there exists simulation function ξ such that

$$\xi(s^4 d(fx, fy), M(x, y)) \geq 0 \; for \; all \; x, y \in X,$$

q.(3.1) where

$$M(x, y) = max \left\{ d^2(x, y), d(x, fx).d(y, fy), \frac{d^2(x, fy) + d^2(y, fx)}{2} \right\}.$$

Then f has a unique fixed point in X, provided f is b-continuous.

Proof. Follows by choosing $g = f$ in Theorem 2.1 [15] □

Remark 3.1 Corollary 3.1 elongate Theorem 1.2 [15] to b-metric spaces.
Corollary 3.2. Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be two self maps on X. Assume that there exists two continuous functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t \leq \psi(t) \; for \; all \; t > 0$ and $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$ such that

$$\psi(s^4 d(fx, gy) \leq \varphi(M(x, y)), \tag{3.8}$$

where

$$M(x, y) = max \left\{ d^2(x, y), d(x, fx).d(y, gy), \frac{d^2(x, gy) + d^2(y, fx)}{2} \right\}, \; for \; all \; x, y \in X.$$

Then f and g have a common fixed point in X, provided either f or g is b-continuous.

Proof. Follows from Theorem 2.1 by choosing $\xi(s,t) = \varphi(t) - \psi(s)\ for\ all\ t, s \in [0,\infty)$.
Remark 3.2 If $g = f$ and $s = 1$ in Theorem 2.1 then Theorem 1.2 [15] of follows as a corollary. □

The following example verify Theorem 2.1.
Example 3.3 : Let $X = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 6 \text{ if } x, y \in [0,1) \\ 7 + \frac{1}{x+y} \text{ if } x, y \in [1,\infty) \\ \frac{7}{6} \text{ otherwise.} \end{cases}$$

Then clearly d is complete b- metric space with coefficient $s = \frac{6}{5}$. Here we observe that when $x = 4/3, z = 2 \in [1,\infty)\ and\ y \in [0,\infty)$ we have $d(x,z) = 7 + 3/10 = 73/10$ and $d(x,y) + d(y,z) = 7/6 + 7/6 = 14/6$ so that $d(x,z) \neq d(x,y) + d(y,z)$.
Hence the given d is a b-metric space with $s = \frac{6}{5}(s > 1)$ but not a metric .
Now we define $f, g : X \to X$ by

$$fx = \begin{cases} \frac{x}{6} + 2 \text{ if} x \in [0,1) \\ 2x - 1 \text{ if} x \in [1,\infty) \end{cases} \quad and\ gx = \begin{cases} x^2 \ \text{ if } x \in [0,1) \\ \frac{1}{x^2} \text{ if } x \in [1,\infty). \end{cases}$$

Then clearly f and g are b-continuous functions.
Now we define $\xi : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $\xi(t,s) = \lambda s - t$ where $\lambda > 1$ and

$$M(x,y) = max\left\{d^2(x,y), d(x,fx).d(y,gy), \frac{1}{2s}\left[d^2(x,gy) + d^2(y,fx)\right]\right\}$$

Case(I):$x, y \in [0,1)$.
In this case, $fx = \frac{x}{6} + 2 \in [1,\infty), gy = y^2 \in [0,1)$. $d(x,y) = 6$, $d(x,fx) = 7/6, d(y,gy) = 6$ and

$$\begin{aligned} \frac{1}{2s}\left(d^2(x,gy) + d^2(y,fx)\right) &= \left(36 + \frac{49}{36}\right) \cdot 2 \cdot \left(\frac{6}{5}\right) \\ &= \left(\frac{36 \cdot 36 + 49}{36}\right) \cdot 2 \cdot \left(\frac{6}{5}\right) \\ &= \left(\frac{1541}{432}\right). \end{aligned}$$

Therefore $M(x,y) = 6$ and

$$\begin{aligned} s^4 d(fx,gy) &= \left(\frac{6}{5}\right)^4 \cdot \left(\frac{7}{6}\right) \\ &= 2.4192 \end{aligned}$$

.

Now, we consider

$$\begin{aligned}\xi(s^4d(fx,gy),M(x,y)) &= \lambda M(x,y)-s^4d(fx,gy)\\ &= 4\cdot 6-2.4192\\ &= 24-2.4192\\ &= 21.5808\\ &> 0.\end{aligned}$$

Case(II):$x,y\in[1,\infty)$.
In this case, $fx=2x-1\in[1,\infty),\ gy=\frac{1}{y^2}\in[0,1)$,
$d(x,y)=7+\frac{1}{x+y}\geq 7, d(x,fx)=7+\frac{1}{4x-2}\geq\ 7, d(y,gy)=\frac{7}{6}$ and

$$\begin{aligned}\frac{d^2(x,gy)+d^2(y,fx)}{2s} &= \left(\frac{49}{36}+49\right)\cdot\left(\frac{5}{12}\right).\\ &= \frac{5}{12}\cdot\left[\frac{1764+49}{36}\right]\\ &= \frac{9065}{432}\\ &= 20.\end{aligned}$$

Therefore $M(x,y)\geq d(x,y)\geq 7$. Now we consider

$$\begin{aligned}\xi(s^4d(fx,gy),M(x,y)) &= \lambda M(x,y)-s^4d(fx,gy)\\ &= 4\cdot 7-2.4192\\ &= 25.5808\\ &> 0.\end{aligned}$$

Case(III) : $x\in[0,1), y\in[1,\infty)$.
In this case,

$$fx=\frac{x}{6}+2\in[1,\infty)\ , gy=\frac{1}{y^2}\in[0,1)\ d(x,y)=\frac{7}{6}, d(x,fx)=\frac{7}{6}, d(y,gy)=\frac{7}{6}$$

$$\begin{aligned}\frac{d^2(x,gy)+d^2(y,fx)}{2s} &= (36+49)\cdot\frac{5}{12}\\ &= \frac{85\cdot 5}{12}\\ &= \frac{425}{12}\end{aligned}$$

$$\begin{aligned}s^4d(fx,gy) &= \left(\frac{6}{5}\right)^4\cdot\frac{7}{6}\\ &= 2.4192\end{aligned}$$

$$\begin{aligned}\xi(s^4d(fx,gy),M(x,y)) &= \lambda M(x,y) - s^4d(fx,gy)\\ &= 6\cdot 4 - 2.4192\\ &= 21.5808\\ &> 0.\end{aligned}$$

Case(IV) : $x \in [1,\infty),\ y \in [0,1)$.
In this case $fx = 2x-1 \in [1,\infty),\ gy = y^2 \in [0,1),\ d(x,y) = \frac{7}{6},\ d(x,fx) = 7 + \frac{1}{4x-2} \geq 7, d(y,gy) = 6$, and

$$\begin{aligned}\frac{d^2(x,gy)+d^2(y,fx)}{2s} &= \left[\left(\frac{7}{6}\right)^2 + \left(\frac{7}{6}\right)^2\right]\cdot\left(\frac{5}{12}\right)\\ &= \left(\frac{490}{432}\right) = 1.1342\end{aligned}$$

Therefore $M(x,y) \geq d(x,fx) \geq 7$ and $s^4d(fx,gy) = 2.4192$

$$\begin{aligned}\xi(s^4d(fx,gy),M(x,y)) &= \lambda M(x,y) - s^4d(fx,gy)\\ &= 4\cdot 7 - 1.1342\\ &= 26.8658\\ &> 0.\end{aligned}$$

Hence (f,g) is a generalized Z-contraction pair of maps and satisfy all the hypotheses of Theorem 2.1 and $x = 1$ is the unique common fixed point of $f\ and\ g$.
Example 3.2. Let X, d, ξ as an Example 3.1 and define $(f_1, f_i)\ for\ i \geq 2$ by

$$f_1(x) = \begin{cases} x \text{ if } x \in [0,1)\\ \frac{1}{x} \text{ if } x \in [1,\infty)\end{cases} \quad and\ f_i(x) = \begin{cases}\frac{x}{2} + i \text{ if } x \in [0,1)\\ 2x-1 \text{ if } x \in [1,\infty).\end{cases}$$

Then $(f_i, f_i) for\ i \geq 2$ is a generalized quadratic type Z-contraction and the sequence $\{f_i\}$ has a unique common fixed point 1.

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