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Isomorphisms in Some Lie Triple Nilalgebras in Dimension Five

Abdoulaye DEMBEGA
Département de mathématiques, Université Norbert Zongo, BP 376 Koudougou, Burkina

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Corresponding Author:
Abdoulaye DEMBEGA

ABSTRACT

In this paper, we study Lie triple nilalgebras of nilindex four that are not power associative. It is known that in dimension five, these algebras admit four possible tables. Here, we are interested in the conditions on the ground field for the existence of isomorphisms between these classes.

KEYWORDS: Lie triple algebra, nilalgebra, nilindex, nilpotence, power-associativity.

I. INTRODUCTION

A Lie triple or an almost Jordan algebra is a nonassociative commutative algebra satisfying $3(x^2y)x - x^3y - 2x(x(xy)) = 0$. It has been introduced since 1965 by J. M. Osborn throughout **[5]**, **[6]** and authors are still interested in their studies as shown by recent results throughout **[1]**, **[2]**. An algebra is said to be a nilalgebra of nilindex n if for all $a \epsilon A$ we have $a^n = 0$ and there exists $x \epsilon A$ such that $x^{n-1} \neq 0$. In our previous studies **[3]**, we have shown that a Lie triple nilalgebra of nilindex 4 in dimension five admits four possible tables. Here we are interested in isomorphisms between these classes.

II. PRELEMINARIES

We can see in the following examples that Lie triple algebras could as well be either non nilalgebras or nilalgebras.

Example 2.1

Let A be an algebra over a field F with basis $\{s, t\}$ and nonzero multiplication table given by $s^2 = s + t$, $st = ts = \frac{1}{2}t$. Then A is a Lie triple non nilalgebra because there is no power of s which cancels.

Example 2.2

Let's consider the commutative algebra A in dimension 11 which multiplication table in the basis $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}\}$ is given by $b_1^2 = b_4$, $b_1b_2 = b_3$, $b_1b_3 = b_5$, $b_1b_4 = b_7$, $b_1b_5 = -\frac{1}{2}b_9$, $b_1b_9 = b_{11}$, $b_2b_4 = b_6$, $b_2b_7 = b_9$, $b_3b_4 = b_8$, $b_3b_7 = b_{11}$, $b_4^2 = b_{10}$, $b_4b_6 = -2b_{11}$, all not written products being zero. Then A is a Lie triple nilalgebra of nilindex 4.

Indeed, if we consider two elements $x = \sum_{i=1}^{11} x_i\, b_i$ and $y = \sum_{i=1}^{11} y_i\, b_i$ of A, we have:

$$x(x(xy)) = \frac{1}{2}(x_1^2x_2y_1 - x_1^3y_2)b_9 + \left(-x_1y_1x_2x_4 + \frac{1}{2}x_3x_1^2y_1 - \frac{1}{2}x_1^3y_3 + x_1^2x_2y_4\right)b_{11},$$

$$x^2 = 2x_1x_2b_3 + x_1^2b_4 + 2x_1x_3b_5 + 2x_2x_4b_6 + 2x_1x_4b_7 + 2x_3x_4b_8 + (-x_1x_5 + 2x_2x_7)b_9 + x_4^2b_{10} + (-4x_4x_6 + 2x_3x_7 + 2x_1x_9)b_{11},$$

$$(x^2y)x = 0$$

and $x^3y = (-y_1x_1^2x_2 + y_2x_1^3)b_9 + (y_3x_1^3 - 2y_4x_1^2x_2 - y_1x_1^2x_3 + 2y_1x_1x_2x_4)b_{11}$.

Since $3(x^2y)x - x^3y - 2x(x(xy)) = 0$, we can say that A is a Lie triple algebra. Furthermore, since $x(x(xy)) = \frac{1}{2}(x_1^2x_2y_1 - x_1^3y_2)b_9 + \left(-x_1y_1x_2x_4 + \frac{1}{2}x_3x_1^2y_1 - \frac{1}{2}x_1^3y_3 + x_1^2x_2y_4\right)b_{11}$, putting $y = x$ we obtain $x^4 = 0$, then A is a nilalgebra of nilindex 4. Because $x^2x^2 \neq 0$, we can say that A is not power-associative. Moreover we have $x^2x^2, x^2(xy) \epsilon J = \langle b_8, b_{10} \rangle$.

In **[3]**, we have shown that if $dim\, A \leq 6$ then for all $a, b \epsilon A$ we have $R_a^3(b) = 0$. In addition we have the following Lemma

Lemma 2.3 [3]

Every Lie triple nilalgebra of nilindex 4 and dimension 5 satisfies $2 \leq dim\, A^2 \leq 3$.

Proof

Since A is a nilalgebra of nilindex 4, we necessarily have $dim\, A^2 \geq 2$.

Now let's show that $dim\, A^2 \leq 3$.

Since $R_x^3 = 0$, according **[4]** algebra A is nilpotent, that means there is $t \geq 4$ such that $A^t = 0$. We easily see that $A^2 \neq A$. This leads to $dim\, A^2 \leq 4$. Let's proove now that $dim\, A^2 < 4$. Assume that $dim\, A^2 = 4$. Then there should exist $y \epsilon A$ such that $A = Ky + A^2$, that gives $A^2 = Ky^2 + A^3$,and $A = \langle\, y, y^2 \rangle + A^3$. Therefore we have $A^2 = \langle y^2, y^3, y^2 y^2 \rangle\ + A^4$, which gives $A = \langle\, y, y^2, y^3, y^2 y^2 \rangle + \ A^4$. In the same way, we have $A = \langle\, y, y^2, y^3, y^2 y^2 \rangle + \ A^5$, that gives $A^5 = A^4$, because of $A^5 \subseteq A^4$. Since A is nilpotent,
we have $A^5 = A^4 = 0$, which gives $A = alg(y)$ and $dim\, A \leq\ 4$; this is a contradiction. Finally, we have $dim\, A^2 \leq 3$.

Theorem 2.4 [3]

Let's consider a Lie triple nilalgebra A of nilindex 4 and dimension 5.

1. If there is $a\ \epsilon\ A$ such that $alg(a) = \langle a, a^2, a^3, a^2 a^2 \rangle$ then there exists $b_0 \epsilon\ A -\ alg(a)$ such that $ab_0 = 0$, $a^2 b_0 = \alpha_1\, a^3 + \alpha_2\, a^2 a^2$ and $b_0^2 = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^2 a^2$.
2. Assume that A does not contain any element x such that $dim(alg\ (x)) = 4$
   i. If $dim\, A^2 = 3$ and $dim\, A^3 = 1$ then there are $x, y\ \epsilon\ A$ such that $A = \langle\, y, yx, x, x^2, x^3 \rangle$ and $(xy)x^2 = \alpha x^3$, $(xy)x = \beta\, x^3$, $(xy)^2 = \gamma\, x^3$, $y(xy) = \lambda\, x^3$ and $y^2 = \epsilon_1(yx) + \epsilon_2\, x^2 + \epsilon_3\, x^3$.
   ii. If $dim\, A^2 = 3\ and\ dim\, A^3 = 2$, then there exist $x_0, y_0 \epsilon\ A$ such that $A \langle y_0, y_0 x_0^2, x_0, x_0^2, x_0^3 \rangle$ and $y_0^2 = \alpha_1\, y_0 x_0^2 + \alpha_2\, x_0^2 + \alpha_3\, x_0^3$.
   iii. If $dim\, A^2 = 2$, then there exist $x_0, y_0, a\ \epsilon\ A$ such that $A = \langle\, x_0, y_0, a, a^2, a^3 \rangle$ and $x_0^2 = \alpha_1\, a^2 + \alpha_2\, a^3$, $y_0^2 = \lambda_1 a^2 + \lambda_2\, a^3$, $x_0 y_0 = \beta_1\, a^2 + \beta_2\, a^3$.

III. CONDITION FOR THE EXISTENCE OF ISOMORPHISMS

The algebra defined in Theorem 2.4 (1) satisfies the following multiplication table when $e_1 = a$, $e_2 = a^2$, $e_3 = a^3$, $e_4 = a^2 a^2$ and $e_5 = b_0$ (the unmentioned products being zero) $e_1^2 = e_2$, $e_1 e_2 = e_3$, $e_2^2 = e_4$, $e_2 e_5 = \alpha_1 e_3 + \alpha_2 e_4$, $e_5^2 = \beta_1 e_2 + \beta_2 e_3 + \beta_3 e_4$.

Let $A(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3)$ be such an algebra.

Proposition 3.1

Two algebras $A(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3)$and $A(\alpha_1{'}, \alpha_2{'}; \beta_1{'}, \beta_2{'}, \beta_3{'})$are isomorphic if and only if $\alpha_i(K^*) = \alpha_i{'}(K^*)$ where $i = 1,2$, $\beta_i(K^*)^2 = \beta_i'(K^*)^2$ where $i = 1,3$ and $\beta_2(K^*) = \beta_2'(K^*)$.

Proof.

Let σ be an isomorphism between two algebras $A(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3)$ and $A(\alpha_1{'}, \alpha_2{'}; \beta_1{'}, \beta_2{'}, \beta_3{'})$. Let $\sigma(e_i) = \sum_{j=1}^{5} a_{ij} e_j$ Then we have

$$\sigma(e_2) = \sigma(e_1)^2 = (a_{11}^2 + \beta_1' a_{15}^2) e_2 + (2a_{11}a_{12} + a_{12}a_{15}\alpha_1' + a_{15}^2 \beta_2') e_3 + (a_{12}^2 + 2a_{12}a_{15}\alpha_2' + a_{15}^2 \beta_3') e_4$$

This implies $a_{21} = a_{25} = 0$, $a_{22} = a_{11}^2 + \beta_1' a_{15}^2$, $a_{23} = 2a_{11}a_{12} + 2a_{11}a_{15}\alpha_1' + a_{15}^2 \beta_2'$ and $a_{24} = a_{12}^2 + 2a_{12}a_{15}\alpha_2' + a_{15}^2 \beta_3'$.

Since the structure constants are independent of the isomorphism $\sigma$and therefore of the a_{ij}, this system implies $a_{15} = 0$. Thus we have

$\sigma(e_2) = a_{11}^2 e_2 + 2a_{11}a_{12} e_3 + a_{12}^2 e_4$ and $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4$.

We also have $\sigma(e_4) = \sigma(e_2)^2 = a_{11}^4 e_4$,

which implies that $a_{11} \neq 0$ otherwise $\sigma(e_4)$ would be zero.

Since $\sigma(e_2) = a_{11}^2 e_2 + 2a_{11}a_{12} e_3 + a_{12}^2 e_4$ and $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4$, we have $\sigma(e_3) = \sigma(e_1)\sigma(e_2) = a_{11}^3 e_3 + a_{11}^2 a_{12} e_4$.

Furthermore, we have

$$0 = \sigma(e_1)\sigma(e_5) = a_{11}a_{51}e_2 + (a_{11}a_{52} + a_{51}a_{22} + a_{12}a_{55})e_3 + (a_1$$

Since the structure constants are independent of the isomorphism σ, this equality implies

$a_{11}a_{51} = 0$, $a_{11}a_{52} + a_{51}a_{22} = 0$, $a_{12}a_{55} = 0$, and $a_{12}a_{52} = 0$. Since $a_{11} \neq 0$, solving this system gives

$a_{51} = a_{52} = a_{12}a_{55} = 0$, **(1)**

and $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5$.

Since $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4$, $\sigma(e_2) = a_{11}^2 e_2 + 2a_{11}a_{12} e_3 + a_{12}^2 e_4$, $\sigma(e_3) = a_{11}^3 e_3 + a_{11}^2 a_{12} e_4$, $\sigma(e_4) = a_{11}^4 e_4$ et $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5$, the determinant of σ is $det\sigma = a_{55}a_{11}^{10} \neq 0$. So $a_{55} \neq 0$ and **(1)** leads to $a_{12} = 0$. This gives $\sigma(e_2) = a_{11}^2 e_2$.

We have

$$\alpha_1 \sigma(e_3) + \alpha_2 \sigma(e_4) = \sigma(e_2)\sigma(e_5) = a_{11}^2 a_{55}(\alpha_1' e_3 + \alpha_2' e_4),$$

which gives $\alpha_1 a_{11}^3 e_3 + \alpha_2 a_{11}^4 e_4 = a_{11}^2 a_{55}(\alpha_1' e_3 + \alpha_2' e_4)$ that means

$\alpha_1 a_{11}^3 = \alpha_1' a_{11}^2 a_{55}$ and $\alpha_2 a_{11}^4 = \alpha_2' a_{11}^2 a_{55}$.

This gives

$$\beta_1 \sigma(e_2) + \beta_2 \sigma(e_3) + \beta_3 \sigma(e_4) = \sigma(e_5)^2 = \beta_1' a_{55}^2 e_2 + \beta_2' a_{55} e_3 + \beta_3' a_{55}^2 e_4$$

which means

$\beta_1 a_{11}^2 e_2 + \beta_2 a_{11}^3 e_3 + \beta_3 a_{11}^4 = \beta_1^{'} a_{55}^2 e_2 + \beta_2^{'} a_{55} e_3 + \beta_3^{'} a_{55}^2 e_4,$

then $\beta_1 a_{11}^2 = \beta_1^{'} a_{55}^2$

$\beta_2 a_{11}^3 = \beta_2^{'} a_{55}$ and $\beta_3 a_{11}^4 = \beta_3^{'} a_{55}^2.$

Corollary 3.2

The following algebras are isomorphism classes : $A(0,0;0,0,0), A(1,0;0,0,0),\ A(0,1;0,0,0),\ A(0,0;0,1,0)$ and $A(0,0;\beta_1,0,\beta_3), with\ (\beta_1,\beta_3) \neq (0,0).$

Proof.

According to the proof of Proposition 3.1, we have $\alpha_1^{'} = a_{11}\, a_{55}^{-1}\alpha_1, \alpha_2^{'} = a_{11}^2 a_{55}^{-1}\alpha_2$

$\beta_1^{'} = a_{11}^2 a_{55}^{-2}\beta_1,\ \beta_2^{'} = a_{11}^3 a_{55}^{-1}\beta_2$ and $\beta_3^{'} = a_{11}^4 a_{55}^{-2}\beta_3.$

Let us consider the following cases:

- If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$ so we have the isomorphism class $A(0,0;0,0,0)$.
- If $\alpha_1 \neq 0\ and\ \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$ then we can choose the isomorphism such that $a_{11} a_{55}^{-1} = \alpha_1^{-1}$. In this case, we have $A(\alpha_1, 0; 0,0,0) \simeq A(1,0;0,0,0)$
- If $\alpha_2 \neq 0$ and $\alpha_1 = \beta_1 = \beta_2 = \beta_3 = 0$ then we can choose the isomorphism such that $a_{11}^2 a_{55}^{-1} = \alpha_2^{-1}$. In this case, we have $A(0, \alpha_2\,; 0,0,0) \simeq A(0,1;0,0,0)$
- If $\beta_2 \neq 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_3 = 0$ then we can choose the isomorphism such that
- $a_{11}^3 a_{55}^{-1} = \beta_2^{-1}$. In this case, we have $A(0,0;0\beta_2,0) \simeq A(0,0;0,1,0)$
- If $\beta_1 \neq 0$ or $\beta\_3 \neq 0$ and $\alpha_1 = \alpha_2 = \beta_2 = 0$ then we have $A(0,0;\beta_1,0,\beta_3) \simeq A(0,0;\beta_1',0,\beta_3')$.

We thus have the isomorphism classes of the corollary.

The algebra defined in Theorem 2.4 $(2)(i)$ satisfies the following multiplication table when $e_1 = x, e_2 = x^2, e_3 = x^3,\ e_4 = xy$ and $e_5 = y$ (the unmentioned products being zero) $e_1^2 = e_2, e_1 e_2 = e_3, e_2^2 = e_3, e_1 e_4 = \alpha\, e_3, e_2 e_4 = \beta\, e_3,$ $e_4^2 = \Upsilon\, e_3,\ e_1 e_5 = e_4,\ e_4 e_5 = \lambda\, e_3,\ e_5^2 = \epsilon_1\, e_2 + \epsilon_2\, e_3 + \epsilon_3\, e_4.$

Let $A(\alpha, \beta, \Upsilon, \lambda; \epsilon_1, \epsilon_2, \epsilon_3\ ),$ be such an algebra.

Proposition 3.3

Two algebras $A(\alpha, \beta, \Upsilon, \lambda; \epsilon_1, \epsilon_2, \epsilon_3\ )$ and $A(\alpha', \beta', \Upsilon', \lambda'; \epsilon_1{'}, \epsilon_2{'}, \epsilon_3{'}\ )$are isomorphic if and only if $\alpha(K^*) = \alpha'(K^*), \beta(K^*) = \beta'(K^*), \epsilon_{3(K^*)} = \epsilon_3{'}(K^*), \lambda(K^*)^2 = \lambda'(K^*)^2,\ \Upsilon(K^*)^2 = \Upsilon'(K^*)^2, \epsilon_i(K^*)^2 = \epsilon_i'(K^*)^2, i = 1,2.$

Proof

Let σbe an isomorphism between two algebras $A(\alpha, \beta, \Upsilon, \lambda; \epsilon_1, \epsilon_2, \epsilon_3\ )$ and $A(\alpha', \beta', \Upsilon', \lambda'; \epsilon_1{'}, \epsilon_2{'}, \epsilon_3{'}\ )$. Let $\sigma(e_i) = \sum_{j=1}^{5} a_{ij} e_j.$ Then we have

$\sigma(e_2) = \sigma(e_1)^2 = (a_{11}^2 + a_{15}^2 \epsilon_1')e_2 + (a_{12}^2 + 2a_{11}a_{12} + 2\alpha' a\_11 a_{14}$

$a_{14}^2 \Upsilon' + 2a_{14}a_{15}\lambda' + a_{15}^2 \epsilon_2^{'})e_3\ + (2a_{11}a_{15} + a_{15}^2 \epsilon_3^{'})e_4.$

This implies $a_{21} = a_{25} = 0,$

$a_{22} = a_{11}^2 + a_{15}^2 \epsilon_1^{'},$

$a_{23} = a_{12}^2 + 2a_{11}a_{12} + 2\alpha^{'} a_{11}a_{14} + 2\beta^{'} a_{12}a_{14} + a_{14}^2 \Upsilon' + 2a_{14}a_{15}\lambda' + a_{15}^2 \epsilon_2^{'}$

and

$a_{24} = 2a_{11}a_{15} + a_{15}^2\ \epsilon_3{'}.$

Since the structure constants of the algebra are independent of the a_{ij} then we have $a_{14} = a_{15} = 0$; and $a_{24} = 0.$ And finally $\sigma(e_2) = a_{11}^2 e_2 + a_{12}(a_{12} + 2a_{11})e_3.$

We also have

$\sigma(e_3) = \sigma(e_2)^2 = a_{11}^4 e_3,$ **(2)**

which implies $a_{11} \neq 0.$

Furthermore, we have

$\sigma(e_3) = \sigma(e_1)\sigma(e_2) = a_{11}^2(a_{11} + a_{12})e_3,$ **(3)**

which means, $a_{11} + a_{12} = a_{11}^2.$

In addition

$\alpha\sigma(e_3) = \sigma(e_1)\sigma(e_4) = a_{11}a_{41}e_2 + (a_{11}a_{42} + a_{12}a_{41}\ + a_{12}a_{42} + \alpha' a_{11}a_{44} + \beta' a_{12}a_{44})e_3 + a_{11}a_{45}e_4,$

which means, $a_{11}a_{41} = 0,$ $\alpha\, a_{11}^4 = a_{11}a_{42} + a_{12}a_{41} + a_{12}a_{42} + \alpha' a_{11}a_{44} + \beta' a_{12}a_{44},\ \ and\ a_{11}a_{45} = 0.$

This implies $a_{41} = a_{45} = 0$ and $a_{42}(a_{11} + a_{12}) = a_{42}a_{11}^2 = 0$ which means $a_{42} = 0.$ And then, $\sigma(e_4) = a_{43}e_3 + a_{44}e_4.$

Then we have equality

$\beta\ \sigma(e_3) = \sigma(e_2)\sigma(e_4) = a_{11}^2 a_{44}\beta' e_3,$

which leads to $\beta a_{11}^4 = \beta' a_{11}^2 a_{44}.$

Furthermore we have $\Upsilon\sigma(e_3) = \sigma(e_4)^2 = a_{44}^2 \Upsilon' e_3,$ and $\Upsilon a_{11}^2 = a_{44}^2 \Upsilon',$

which leads to, $a_{45} = 0$ and $\Upsilon\, a_{11}^2 = \Upsilon' a_{44}^2.$ We rewrite $\sigma(e_4) = a_{43}e_3 + a_{44}e_4.$

We also have

$\epsilon_1\sigma(e_2) + \epsilon_2\sigma(e_3) + \epsilon_3\sigma(e_4) = \sigma(\ e_5\ )^2 = (a_{51}^2 + \epsilon_1' a_{55}^2)e_2 + (2a_{51}a_{52} + a_{52}^2 + 2\alpha' a_{51}a_{54} + 2\beta' a_{52}a_{54} + \Upsilon' a_{54}^2 + 2\lambda' a_{54}a_{55} + \epsilon_2' a_{55}^2)e_3 + (2a_{51}a_{55} + \epsilon_3' a_{55}^2)e_4,$

which means

$$\begin{cases} \epsilon_1\, a_{11}^2 = a_{51}^2 + \epsilon_1' a_{55}^2 \\ \epsilon_1 a_{12}(a_{12} + 2a_{11}) + \epsilon_2 a_{11}^4 + \epsilon_3 a_{43} = 2a_{51}a_{52} + a_{52}^2 + \\ 2\alpha' a_{51}a_{54} + 2\beta' a_{52}a_{54} + \Upsilon' a_{54}^2 + 2\lambda' a_{54}a_{55} + \epsilon_2' a_{55}^2 \\ \epsilon_3 a_{44} = 2a_{51}a_{55} + \epsilon_3' a_{55}^2 \end{cases}$$

This system implies $a_{51} = a_{52} = 0$ and $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5$

We also have, $\sigma(e_4) = \sigma(e_1)\sigma(e_5) = (a_{11}a_{54}\alpha' + a_{12}a_{54}\beta')e_3 + a_{11}a_{55}e_4$

which means $a_{43} = a_{11}a_{54}\alpha' + a_{12}a_{54}\beta'$ and $a_{44} = a_{11}a_{55}$, giving then $a_{43} = a_{54} = 0$. We then obtain $\sigma(e_4) = a_{44}e_4$ et $\sigma(e_5) = a_{53}e_3 + a_{55}e_5$.

On the other hand, we have $\lambda\sigma(e_3) = \sigma(e_4)\sigma(e_5) = a_{44}a_{55}\lambda' e_3$ which gives $a_{11}^4\lambda = a_{44}a_{55}\lambda'$.

In summary, we have $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3$, $\sigma(e_2) = a_{11}^2e_2 + a_{12}(a_{12} + 2a_{11})e\_3$, $\sigma(e_3) = a_{11}^2(a_{11} + a_{12})e_3 = a_{11}^4e_3$, $\sigma(e_4) = a_{44}e_4$ and $\sigma(e_5) = a_{53}e_3 + a_{55}e_5$. Let us recall that $a_{12} = a_{11}^2 - a_{11} = a_{11}(a_{11} - 1)$ according to **(2), (3)** and $a_{44} = a_{11}a_{55}$. Since $a_{11} \neq 0$, in order to cancel a_{12}, let's choose $a_{11} = 1$. The determinant of σ is $\det(\sigma) = a_{55}^2 \neq 0$. This leads to $a_{55} \neq 0$.

Expressions in the system are simplified and we ultimately have

$\alpha = a_{55}\,\alpha', \beta = a_{55}\beta',\ \lambda = a_{55}^2\,\lambda',\ \Upsilon = \Upsilon' a_{55}^2, \epsilon_1 = a_{55}^2\epsilon_1',$
$\epsilon_2 = \epsilon_2' a_{55}^2$

and $\epsilon_3 = \epsilon_3' a_{55}$.

Corollary 3.4

The following algebras are isomorphism classes : $A(0{,}0{,}0{,}0; 0{,}0{,}0)$, $A(1{,}0{,}0{,}0; 0{,}0{,}0)$, $A(0{,}1{,}0{,}0; 0{,}0{,}0)$, $A(0{,}0{,}0{,}0; 0{,}0{,}1)$ and $A(0{,}0, \Upsilon, \lambda; \epsilon_1, \epsilon_2, 0)$, with $(\Upsilon, \lambda, \epsilon_1, \epsilon_2) \neq (0{,}0{,}0{,}0)$.

Proof

Indeed according to the proof of Proposition 3.3 we have $\epsilon_1 = \epsilon_1' a_{55}^2, \epsilon_2 = \epsilon_2' a_{55}^2$, $\epsilon_3 = \epsilon_3' a_{55}$, $\alpha = \alpha' a_{55}$, $\beta = \beta' a_{55}$, $\lambda = \lambda' a_{55}^2$ and $\Upsilon = \Upsilon' a_{55}^2$.

Let's consider the following cases

- If $\alpha = \beta = \Upsilon = \lambda = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ so we have the isomorphism class $A(0{,}0{,}0{,}0; 0{,}0{,}0)$.
- If $\alpha \neq 0$ and $\beta = \Upsilon = \lambda = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ we can then choose the isomorphism such that $\alpha = a_{55}$. Then we have $A(\alpha, 0{,}0{,}0; 0{,}0{,}0) \simeq A(1{,}0{,}0{,}0; 0{,}0{,}0)$.
- If $\beta \neq 0$ and $\alpha = \Upsilon = \lambda = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ we can then choose the isomorphism such that $\beta = a_{55}$. Then we have $A(0, \beta, 0{,}0; 0{,}0{,}0) \simeq A(0{,}1{,}0{,}0; 0{,}0{,}0)$.
- If $\epsilon_3 \neq 0$ and $\alpha = \beta = \Upsilon = \lambda = \epsilon_1 = \epsilon_2 = 0$ we can then choose the isomorphism such that $\epsilon_3 = a_{55}$. So we have $A(0{,}0{,}0{,}0; 0{,}0, \epsilon_3) \simeq A(0{,}0{,}0{,}0; 0{,}0{,}1)$.
- If $(\Upsilon, \lambda, \epsilon_1, \epsilon_2) \neq (0{,}0{,}0{,}0)$ then $A(0{,}0, \Upsilon, \lambda; \epsilon_1, \epsilon_2, 0) \simeq A(0{,}0, \Upsilon', \lambda'; \epsilon_1', \epsilon_2', 0)$.

We then have the isomorphism classes of the corollary.

The algebra defined in Theorem 2.4 (2)(ii) satisfies the following multiplication table when $e_1 = x_0$, $e_2 = x_0^2$, $e_3 = x_0^3$, $e_4 = y_0x_0^2$ and $e_5 = y_0$ (the unmentioned products being zero) $e_1^2 = e_2$, $e_1e_2 = e_3$, $e_2^2 = e_3$, $e_2e_5 = e_4, e_5^2 = \alpha_1e_2 + \alpha_2e_3 + \alpha_3e_4$.

Let $A(\alpha_1, \alpha_2, \alpha_3)$, be such an algebra.

Proposition 3.5

Two algebras $A(\alpha_1, \alpha_2, \alpha_3)$ and $A(\alpha_1', \alpha_2', \alpha_3')$ are isomorphic if and only if $\alpha_i(K^*)^2 = \alpha_i'(K^*)^2$, $i = 1{,}2$ and $\alpha_3(K^*) = \alpha_3'(K^*)$

Proof

Let σ be an isomorphism between $A(\alpha_1, \alpha_2, \alpha_3)$ and $A(\alpha_1', \alpha_2', \alpha_3')$. Let $\sigma(e_i) = \sum_{j=1}^{5} a_{ij}e_j$. We have

$\sigma(e_2) = \sigma(e_1)^2 = (a_{11}^2 + \alpha_1' a_{15}^2)e_2 + (2a_{11}a_{12} + a_{12}^2 + a_{15}^2\alpha_2')e_3$
$+(2a_{12}a_{15} + \alpha_3' a_{15}^2)e_4$.

This gives $a_{21} = a_{25} = 0$, $a_{22} = a_{11}^2 + \alpha_1' a_{15}^2$, $a_{23} = 2a_{11}a_{12} + a_{12}^2 + a_{15}^2\alpha_2'$ and $a_{24} = 2a_{12}a_{15} + \alpha_3' a_{15}^2$. Since the structure constants are independent of the a_{ij}, then

$a_{15} = 0$ and $\sigma(e_2) = a_{11}^2e_2 + a_{12}(a_{12} + 2a_{11})e_3$.

We also have

$\sigma(e_3) = \sigma(e_2)^2 = a_{11}^4e_3$ **(4)**

which implies $a_{11} \neq 0$.

Furthermore, we have

$\sigma(e_3) = \sigma(e_1)\sigma(e_2) = a_{11}^2(a_{11} + a_{12})e_3$, **(5)**

which means, $a_{11} + a_{12} = a_{11}^2$.

We also have $\sigma(e_4) = \sigma(e_2)\sigma(e_5) = a_{11}^2(a_{51} + a_{52})e_3 + a_{11}^2a_{55}e_4$ which leads to $a_{41} = a_{42} = a_{45} = 0$, $a_{43} = a_{11}^2(a_{51} + a_{52})$ and $a_{44} = a_{11}^2a_{55}$.

Then we can write $\sigma(e_4) = a_{11}^2(a_{51} + a_{52})e_3 + a_{11}^2a_{55}e_4$.

We have $0 = \sigma(e_1)\sigma(e_5) = a_{11}a_{51}e_2 + (a_{11}a_{52} + a_{51}a_{12} + a_{12}a_{52})e_3 + a_{55}a_{12}e_4$

which means $a_{11}a_{51} = 0$, $a_{11}a_{52} + a_{51}a_{12} + a_{12}a_{52} = 0$ and $a_{55}a_{12} = 0$. This implies that $a_{51} = 0$, $0 = a_{52}(a_{11} + a_{12}) = a_{52}a_{11}^2$ which gives $a_{52} = 0$, and $a_{12}a_{55} = 0$.**(6)**

Then we can write $\sigma(e_4) = a_{11}^2 a_{55} e_4$ and $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5$ which implies $a_{55} \neq 0$. So (6) leads to $a_{12} = 0$ and the expression of $\sigma(e_2)$ becomes $\sigma(e_2) = a_{11}^2 e_2$.

On the other hand, we have $\alpha_1\sigma(e_2) + \alpha_2\sigma(e_3) + \alpha_3\sigma(e_4) = \sigma(e_5)^2 = a_{55}^2(\alpha_1' e_2 + \alpha_2' e_3 + \alpha_3' e_4)$,

which means $\alpha_1 a_{11}^2 = \alpha_1' a_{55}^2$, $\alpha_2 a_{11}^4 = \alpha_2' a_{55}^2$ and $\alpha_3 a_{11}^2 = \alpha_3' a_{55}$.

Because of (4) and (5), we have $\sigma(e_3) = a_{11}^3 e_3 = a_{11}^4 e_3$, which means $a_{11} = 1$.

Finally we have $\alpha_1 = \alpha_1' a_{55}^2$, $\alpha_2 = \alpha_2' a_{55}^2$ and$\alpha_3 = \alpha_3' a_{55}$.

Corollary 3.6

The following algebras are isomorphism classes : $A(0,0,0)$, $A(0,0,1)$ et $A(\alpha_1,\alpha_2,0)$, avec $(\alpha_1,\alpha_2) \neq (0,0)$.

Proof

Indeed, because of the proof of Proposition 3.5 we have $\alpha_1' = a_{55}^{-2}\alpha_1$, $\alpha_2' = a_{55}^{-2}\alpha_2$ and $\alpha_3' = a_{55}^{-1}\alpha_3$.

Let's consider the following cases:

- If $\alpha_1 = \alpha_2 = \alpha_3 = 0$ so we have the isomorphism class $A(0,0,0)$
- If $\alpha_3 \neq 0$ we can then choose the isomorphism such that $a_{55} = \alpha$
- If $(\alpha_1,\alpha_2) \neq (0,0)$ and $\alpha_3 = 0$ then we have $A(\alpha_1,\alpha_2,0) \simeq A(\alpha_1',\alpha_2',0)$.

We thus have the isomorphism classes of the corollary.

The algebra defined in Theorem 2.4 (2)(iii) satisfies the following multiplication table when $e_1 = a$, $e_2 = a^2$, $e_3 = a^3$, $e_4 = x_0$ and $e_5 = y_0$ (the unmentioned products being zero) $e_1^2 = e_2$, $e_1 e_2 = e_3$, $e_2^2 = e_3$, $e_4^2 = \alpha_1 e_2 + \alpha_2 e_3$, $e_4 e_5 = \beta_1 e_2 + \beta_2 e_3$, $e_5^2 = \lambda_1 e_2 + \lambda_2 e_3$.

Let $A(\alpha_1,\alpha_2;\beta_1,\beta_2;\lambda_1,\lambda_2)$, be such an algebra.

Proposition 3.7

Two algebras $A(\alpha_1,\alpha_2;\beta_1,\beta_2;\lambda_1,\lambda_2)$ and $A(\alpha_1',\alpha_2';\beta_1',\beta_2';\lambda_1',\lambda_2')$ are isomorphic if and only if $\alpha_i'(K^*)^2 = \alpha_i(K^*)^2$, $\beta_i'(K^*) = \beta_i(K^*)$ and $\lambda_i'(K^*)^2 = \lambda_i(K^*)^2$ with $i = 1,2$.

Proof

Let σ be an isomorphism between $A(\alpha_1,\alpha_2;\beta_1,\beta_2;\lambda_1,\lambda_2)$ and $A(\alpha_1',\alpha_2';\beta_1',\beta_2';\lambda_1',\lambda_2')$. Let $\sigma(e_i) = \sum_{j=1}^{5} a_{ij} e_j$. We have

$$\sigma(e_2) = \sigma(e_1)^2 = (a_{11}^2 + \alpha_1' a_{11}^2 + \lambda_1' a_{15}^2 + 2a_{14}a_{15}\beta_1')e_2$$
$$+(2a_{12}a_{11} + a_{12}^2 + a_{14}^2\alpha_2' + a_{15}^2\lambda_2' + 2a_{14}a_{15}\beta_2')e_3,$$

which gives $a_{21} = a_{24} = a_{25} = 0$,

$a_{22} = a_{11}^2 + \alpha_1' a_{11}^2 + \lambda_1' a_{15}^2 + 2a\_14a\_15\beta_1'$ and $a_{23} = 2a_{12}a_{11} + a_{12}^2 + a_{14}^2\alpha_2' + a_{15}^2\lambda_2' + 2a_{14}a_{15}\beta_2'$,

which means $a_{14} = a_{15} = 0$. Then we have $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3$ and
$\sigma(e_2) = a_{11}^2 e_2 + (a_{12}^2 + 2a_{11})e_3$.

We also have $\sigma(e_3) = a_{11}^4 e_3$ which implies $a_{11} \neq 0$ and $\sigma(e_3) = \sigma(e_1)\sigma(e_2) = a_{11}^2(a_{11} + a_{12})e_3$ and leads to $a_{11} + a_{12} = a_{11}^2$.

Furthermore we have $0 = \sigma(e_1)\sigma(e_4) = a_{11}a_{41}e_2 + (a_{11}a_{42} + a_{41}a_{12} + a_{12}a_{42})e_3$

which gives $a_{11}a_{41} = 0$

and $a_{11}a_{42} + a_{41}a_{12} + a_{12}a_{42} = 0$. That means $a_{41} = 0$ and $0 = a_{42}(a_{11} + a_{12}) = a_{42}a_{11}^2$, then we have $a\_42 = 0$. We can also write
$\sigma(e_4) = a_{43}e_3 + a_{44}e_4 + a_{45}e_5$.

On the other hand, we have

$$\alpha_1\sigma(e_2) + \alpha_2\sigma(e_3) = \sigma(e_4)^2 = \left(a_{44}^2\alpha_1' + 2a_{44}a_{45}\beta_1' + a_{45}^2\lambda_1'\right)e_2$$
$$+(a_{44}^2\alpha_2' + 2a_{44}a_{45}\beta_2' + a_{45}^2\lambda_2')e_3,$$

which means

$$\alpha_1 a_{11}^2 e_2 + [\alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4]e_3 =$$
$$(a_{44}^2\alpha_1' + 2a_{44}a_{45}\beta_1' + a_{45}^2\lambda_1')e_2$$
$$+(a_{44}^2\alpha_2' + 2a_{44}a_{45}\beta_2' + a_{45}^2\lambda_2')e_3,$$

and

$$(S_1)\begin{cases} \alpha_1 a_{11}^2 = a_{44}^2\alpha_1{}' + 2a_{44}a_{45}\beta_1' + a_{45}^2\lambda_1', \\ \quad \alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4 = \\ \quad\quad a_{44}^2\lambda_2' + 2a_{44}a_{45}\beta_2' + a_{45}^2\lambda_2'. \end{cases}$$

We also have

$$0 = \sigma(e_1)\sigma(e_5) = a_{11}a_{51}e_2 + (a_{11}a_{52} + a_{51}a_{12} + a_{12}a_{52})e_3,$$

which means $a_{11}a_{51} = 0$ and

$a_{11}a_{52} + a_{51}a_{12} + a_{12}a_{52} = 0$. That leads to $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5$.

In addition

$$\beta_1\sigma(e_2) + \beta_2\sigma(e_3) = \sigma(e_4)\sigma(e_5) =$$
$$(a_{44}a_{54}\alpha_1' + (a_{44}a_{55} + a_{54}a_{45})\beta_1' + a_{45}a_{55}\lambda_1')e_2$$

gives,

$$\beta_1 a_{11}^2 e_2 + [\beta_1(a_{12}^2 + 2a_{12}a_{11}) + \beta_2 a_{11}^4]e_3 =$$
$$(a_{44}a_{54}\alpha_1' + (a_{44}a_{55} + a_{54}a_{45})\beta_1' + a_{45}a_{55}\lambda_1')e_2$$
$$+(a_{44}a_{54}\alpha_2' + (a_{44}a_{55} + a_{54}a_{45})\beta_2' + a_{45}a_{55}\lambda_2')e_3,$$

Then

$$(S_2)\begin{cases} \beta_1 a_{11}^2 = a_{44}a_{54}\alpha_1' + (a_{44}a_{55} + a_{54}a_{45})\beta_1' + a_{45}a_{55}\lambda_1' \\ \quad \beta_1(a_{12}^2 + 2a_{12}a_{11}) + \beta_2 a_{11}^4 = a_{44}a_{54}\alpha_2' + \\ \quad\quad (a_{44}a_{55} + a_{54}a_{45})\beta_2' + a_{45}a_{55}\lambda_2' \end{cases}$$

Furthermore we have,

$$\lambda_1\sigma(e_2) + \lambda_2\sigma(e_3) = \sigma(e_5)^2 =$$

$(a_{54}^2\lambda_1' + 2a_{54}a_{55}\beta_1' + a_{55}^2\lambda_1')e_2$
$+(a_{54}^2\lambda_2' + 2a_{54}a_{55}\beta_2' + a_{55}^2\lambda_2')e_3,$

which means,

$\lambda_1 a_{11}^2 e_2 + [\lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4]e_3 =$
$(a_{54}^2\lambda_1' + 2a_{54}a_{55}\beta_1' + a_{55}^2\lambda_1')e_2$
$+(a_{54}^2\lambda_2' + 2a_{54}a_{55}\beta_2' + a_{55}^2\lambda_2')e_3,$

then

$$(S_3)\begin{cases} \lambda_1 a_{11}^2 = a_{54}^2\lambda_1' + 2a_{54}a_{55}\beta_1' + a_{55}^2\lambda_1', \\ \lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4 = a_{54}^2\lambda_2' + 2a_{54}a_{55}\beta_2' + a_{55}^2\lambda_2' \end{cases}$$

Then summary, we have $\sigma(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3,$ $\sigma(e_2) = a_{11}^2 e_2 + a_{12}(a_{12} + 2a_{11})e_3, \sigma(e_3) = a_{11}^4 e_3,$ $\sigma(e_4) = a_{43}e_3 + a_{44}e_4 + a_{45}e_5$ and $\sigma(e_5) = a_{53}e_3 + a_{54}e_4 + a_{55}e_5.$

The determinant of σ is $det\sigma = a_{11}^7(a_{44}a_{55} - a_{54}a_{45}) \neq 0.$ Let $\delta = a_{44}a_{55} - a_{54}a_{45}.$ Since $det\sigma = a_{11}^7\delta \neq 0,$ we have $\delta \neq 0.$

Using the first lines of systems $(S_1), (S_2)$ and (S_3) we can define the following system

$$(S)\begin{cases} \alpha_1 a_{11}^2 = a_{44}^2\alpha_1' + 2a_{44}a\_45\beta_1' + a_{45}^2\lambda_1', \\ \beta_1 a_{11}^2 = a_{44}a_{54}\alpha_1' + (a_{44}a_{55} + a_{54}a_{45})\beta_1' + a_{45}a_{55}\lambda_1', \\ \lambda_1 a_{11}^2 = a_{54}^2\lambda_1' + 2a_{54}a_{55}\beta_1' + a_{55}^2\lambda_1' \end{cases}$$

in which α_1', β_1' and λ_1' are the variables.

In the same way, using the second lines, we define the system (S')

$$')\begin{cases} \alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4 = \\ a_{44}^2\alpha_2' + 2a_{44}a_{45}\beta_2' + a_{45}^2\lambda_2', \\ \beta_1(a_{12}^2 + 2a_{11}a_{12}) + \beta_2 a_{11}^4 = a_{44}a_{54}\alpha_2' \\ +(a_{44}a_{55} + a_{54}a_{45})\beta_2' + a_{45}a_{55}\lambda_2' \\ \lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4 = a_{54}^2\lambda_2' + \\ 2a_{54}a_{55}\beta_2' + a_{55}^2\lambda_2', \end{cases}$$

in which α_2', β_2' and λ_2' are variables.

The systems (S) and (S') admits the same the same determinant which is

$\Delta = (a_{44}a_{55} - a_{54}a_{45})^3 = \delta^3 \neq 0.$

Then these are two Cramer systems. Let's first solve (S). It gives

$$\alpha_1' = \frac{\Delta_{\alpha_1'}}{\Delta} = \frac{a_{11}^2 a_{55}^2}{\delta^2}\alpha_1 - 2\frac{a_{11}^2 a_{45}a_{55}}{\delta^2}\beta_1 + \frac{a_{11}^2 a_{45}^2}{\delta^2}\lambda_1,$$

$$\beta_1' = \frac{\Delta_{\beta_1'}}{\Delta} = -\frac{a_{11}^2 a_{54}a_{55}}{\delta^2}\alpha_1 + \frac{a_{11}^2(a_{44}a_{55} + a_{54}a_{45})}{\delta^2}\beta_1$$
$$-\frac{a_{11}^2 a_{44}a_{45}}{\delta^2}\lambda_1,$$

and

$$\lambda_1' = \frac{\Delta_{\lambda_1'}}{\Delta} = \frac{a_{11}^2 a_{54}^2}{\delta^2}\alpha_1 - 2\frac{a_{11}^2 a_{44}a_{54}}{\delta^2}\beta_1 + \frac{a_{11}^2 a_{44}^2}{\delta^2}\lambda_1 .$$

For (S') we have:

$$\alpha_2' = \frac{\Delta_{\alpha_2'}}{\Delta} = \frac{a_{55}^2}{\delta^2}(\alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4) -$$
$$2\frac{a_{45}a_{55}}{\delta^2}(\beta_1(a_{12}^2 + 2a_{11}a_{12}) + \beta_2 a_{11}^4) +$$
$$\frac{a_{45}^2}{\delta^2}(\lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4),$$

$$\beta_2' = \frac{\Delta_{\beta_2'}}{\Delta} = -\frac{a_{54}a_{55}}{\delta^2}(\alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4) +$$
$$\frac{(a_{44}a_{55} + a_{45}a_{54})}{\delta^2}(\beta_1(a_{12}^2 + 2a_{11}a_{12}) + \beta_2 a_{11}^4) -$$
$$\frac{a_{44}a_{45}}{\delta^2}(\lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4),$$

and

$$\lambda_2' = \frac{\Delta_{\lambda_2'}}{\Delta} = \frac{a_{54}^2}{\delta^2}(\alpha_1(a_{12}^2 + 2a_{11}a_{12}) + \alpha_2 a_{11}^4) -$$
$$2\frac{a_{44}a_{54}}{\delta^2}(\beta_1(a_{12}^2 + 2a_{11}a_{12}) + \beta_2 a_{11}^4) +$$
$$\frac{a_{44}^2}{\delta^2}(\lambda_1(a_{12}^2 + 2a_{11}a_{12}) + \lambda_2 a_{11}^4).$$

Let's take $a_{11} = 1$, then $a_{12} = 0$ and we have:

$$\alpha_2' = \frac{a_{55}^2}{\delta^2}a_{11}^4\alpha_2 - 2\frac{a_{45}a_{55}}{\delta^2}a_{11}^4\beta_2 + \frac{a_{45}^2}{\delta^2}a_{11}^4\lambda_2,$$

$$\beta_2' = -\frac{a_{54}a_{55}}{\delta^2}a_{11}^4\alpha_2 + \frac{(a_{44}a_{55} + a_{45}a_{54})}{\delta^2}a_{11}^4\beta_2$$
$$-\frac{a_{44}a_{45}}{\delta^2}a_{11}^4\lambda_2,$$

and $\lambda_2' = \frac{a_{54}^2}{\delta^2}a_{11}^4\alpha_2 - 2\frac{a_{44}a_{54}}{\delta^2}a_{11}^4\beta_2 + \frac{a_{44}^2}{\delta^2}a_{11}^4\lambda_2.$

In particular for $a_{45} = a_{54} = 0$ we have $\delta = a_{44}a_{55} \neq 0$ and this gives $\alpha_i' = \frac{a_{55}^2}{\delta^2}\alpha_i, \beta_i' = \frac{a_{44}\,a_{55}}{\delta^2}\beta_i$ and $\lambda_i' = \frac{a_{44}^2}{\delta^2}\lambda_i.$

That means $\alpha_i' = a_{44}^{-2}\alpha_i,$ $\beta_i' = a_{44}^{-1}a_{55}^{-1}\beta_i$ and $\lambda_i' = a_{55}^{-2}\lambda_i, for\ i = 1{,}2.$

Corollary 3.8

The following algebras are isomorphism classes : $A(0{,}0;0{,}0;0{,}0),$ $A(0{,}0;1{,}0;0{,}0),$ $A(0{,}0;0{,}1;0{,}0)$ and $A(\alpha_1,\alpha_2;0{,}0;\lambda_1,\lambda_2)\ with\ (\alpha_1,\alpha_2,\lambda_1,\lambda_2) \neq (0{,}0{,}0{,}0).$

Proof

Indeed, because of the proof of Proposition 3.7, we have $\alpha_i' = a_{44}^{-2}\alpha_i, \beta_i' = a_{44}^{-1}a_{55}^{-1}\beta_i$ and $\lambda_i' = a_{55}^{-2}\lambda_i,$ with $i = 1{,}2.$

Let's consider the following cases :

- If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \lambda_1 = \lambda_2 = 0$ so we have the isomorphism class $A(0{,}0;0{,}0;0{,}0)$.
- If $\beta_1 \neq 0$ and $\alpha_1 = \alpha_2 = \beta_2 = \lambda_1 = \lambda_2 = 0$ then we can choose the isomorphism such that $a_{44}a_{55} = \beta_1.$ Then, we have $A(0{,}0;\beta\_1{,}0;0{,}0) \simeq A(0{,}0;1{,}0;0{,}0).$
- If $\beta_2 \neq 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \lambda_1 = \lambda_2 = 0$ then we can choose the isomorphism such that $a_{44}a_{55} = \beta_2.$ Then, we have $A(0{,}0;0,\beta_2;0{,}0) \simeq A(0{,}0;0{,}1;0{,}0).$
- If $(\alpha_1,\alpha_2,\lambda_1,\lambda_2) \neq (0{,}0{,}0{,}0)$ and $\beta_1 = \beta_2 = 0$ then we have $A(\alpha_1,\alpha_2;0{,}0;\lambda_1,\lambda_2) \simeq A(\alpha_1',\alpha_2';0{,}0;\lambda_1',\lambda_2')$

Then we have the isomorphism classes of the corollary.

REFERENCES

1. J. Bayara, A. Konkobo and M. Ouattara, 2017. Equations des algèbres Lie triple qui sont des algèbres train, Indagationes mathematicae 28(2) : 390-405.https://doi.org/10.1016/j.indag.2016.10.004
2. A. Dembega, A. Konkobo and M. Ouattara, 2019. Derivations and dimensionally nilpotent derivations in Lie triple algebras, Gulf Journal of Mathematics, 7(2), 71-84.https://doi.org/10.56947/gjom.v7i2.192
3. A. Dembega, 2019. Algèbres presque de jordan et deux classes d'algèbres commutatives de degré 5, PhD thesis, University Joseph KI-ZERBO, Ouagadougou (Burkina Faso) 34-47.
4. Juan C. Gutiérrez Fernandez, 2009. Commutative Finite-Dimensional Algabras Satisfying x(x(xy))=0 are nilpotent. Comm. Algebra, 37(10), 3760-3776. https://doi.org/10.1080/00927870802502944
5. J.M. Osborn, 1965. Commutative algebras satisfying an identity of degree four, Trans. Amer. Math. Soc.16, 1114-1120. https://doi.org/10.2307/2035628
6. J.M. Osborn, 1965. Identities of non-associative algebras, Can. J. Math.17, 78-92. https://doi.org/10.4153/CJM-1965-008-3
7. R.D. Schafer, 1966. An introduction to nonassociative algebras, Academic Press, New York. https://www.gutenberg.org/files/25156/25156-pdf.pdf