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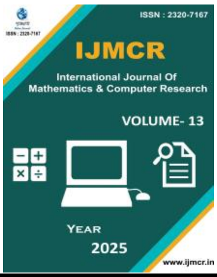

Method of Calculating Area and Arc Length of Sine and Cosine Functions in Polar Coordinates

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ABSTRACT
The region area is the measurement of two-dimensional surface bounded by a curve. The region area is determined using an integral, based on the distance between points rotating according to the rotation angle. The arc length is a measure of the distance along the curved line formed by the curve. The arc length is determined using an integral, based on the distance between points rotating according to the rotation angle. The arc length is the length of the curve. The purpose of this study was to find the area and the arc length of a figure formed by a curve using polar coordinates. The results obtained that the region area be formed by a curve is directly proportional to the square of the curve function, while the arc length on the curve is influenced by the square of the curve function and its first derivative. Simulations were performed on sine and cosine functions.

KEYWORDS: Polar Coordinate, Integral, Region Area, Arc Length, Rotation Angel, Simulation

I. INTRODUCTION

The region area is the measure of a two-dimensional surface bounded by a curve. In calculus with Cartesian coordinates, the region area is determined using integral [3]. In polar coordinates, the abscissa and ordinate are transformed into the distance and rotation angle of a point. Therefore, in polar coordinates, the region area is determined using integrals, based on the distance of the point rotating according to its rotation angle. The area swept by a line from the origin to the point of the curve moving according to its rotation angle will form the area of that region [2]. Meanwhile, the arc length is a measure of the distance along the curved line formed by the curve. The arc length is one-dimensional shape. In polar coordinates, determining the arc length also uses an integral, but based on the distance between points rotating according to their rotation angle. The curve image of points moving along their rotation angle will form a curve. So the arc length is the length of the curve [1].

The purpose of this research is to find the area and the arc length of a shape formed by a curve using polar coordinates. In Cartesian coordinates, the integral is based on the Riemann integral, while in polar coordinates, the integral is based on the radius and rotation angle of the curve [4]. The functions used in this research are sine and cosine functions. The discussion consists of elaboration of the properties of sine and cosine functions in the integral for several constants. The article presentation is in the form of a simulation of several constants about the area and the arc length formed by the curve in polar coordinates.

II. THE AREA OF A REGION AND THE ARC LENGTH OF A CURVE IN POLAR COORDINATE

The area of a region and the length of a curve are two interrelated problems. They form a single unit that forms an interesting picture.

A. The Area of a Region in Polar Coordinate

First, we define the sector area as basis for calculating the area of a region in polar coordinates.

Theorem 1.
The sector is a portion of a circle. The area of a sector A is $A=\frac{1}{2}r^2\theta$, where r is the radius of a circle, θ is the rotation angle of a sector.
Proof:
Note that this comparison

$$\frac{A}{B}=\frac{C}{D} \Leftrightarrow \frac{\text{Sector area}}{\text{Circle area}}=\frac{\text{Sector angle}}{\text{Circle angle}}$$

$$\Leftrightarrow \frac{\text{Sector area}}{\pi r^2}=\frac{\theta}{360}$$

$$\text{Sector area} = \frac{\theta}{360}\pi r^2 = \frac{\theta}{2\pi}\pi r^2 = \tfrac{1}{2}r^2\theta .$$

Therefore, the area of a sector is $A = \tfrac{1}{2}r^2\theta$. (1)

The Polar Area on One Curve

In polar coordinate, the angle θ is considered as an independent variable, while r is considered a dependent variable, which is as a function of θ. So that it be writen by $r = f(\theta)$.

If $D = \{(r,\theta) \in \mathbb{R}^2 \mid \alpha \le \theta \le \beta\}$, then based on Equation (1), the area of a region in polar coordinate be obtained by $A = \tfrac{1}{2}\int_{\theta=\alpha}^{\beta} r^2 d\theta = \tfrac{1}{2}\int_{\theta=\alpha}^{\beta}\left[f(\theta)\right]^2 d\theta$. (2)

The Polar Area Between Two Curves

If there is an area A. It be formed by two graphs. Suppose that area A is outside the first curve $r_1 = f_1(\theta)$ but area A is inside the second curve $r_2 = f_2(\theta)$, then based on Equation (2) be obtained that

$$A = \tfrac{1}{2}\int_{\theta=\alpha}^{\beta}\left(r_2^2 - r_1^2\right)d\theta = \tfrac{1}{2}\int_{\theta=\alpha}^{\beta}\left(f_2^2(\theta) - f_1^2(\theta)\right)d\theta \quad (3)$$

We provide important formulas that are useful for making calculation integral easier:

a. $\sin 2\theta = 2\sin\theta\cos\theta$.

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

B. The Arc Length of a Curve in Polar Coordinate

Below we present the basis for determining the arc length of a curve in polar coordinate. The arc length of a curve in polar coordinates is written in Theorem 2.

Theorem 2.

Let $r = f(\theta)$ be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The arc length of the curve $r = f(\theta)$, with the rotation angle $\alpha \le \theta \le \beta$ is

$$L = \int_{\theta=\alpha}^{\beta}\left(r^2 + (r')^2\right)^{\frac{1}{2}}d\theta . \quad (4)$$

Proof:

Let $x = r\cos\theta$ and $y = r\sin\theta$, so that $x' = r'\cos\theta - r\sin\theta$ and $y' = r'\sin\theta + r\cos\theta$. Then $(x')^2 + (y')^2 = r^2 + (r')^2$, and

$$dL = \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{1}{2}} = \left[(x')^2 + (y')^2\right]^{\frac{1}{2}} = \left(r^2 + (r')^2\right)^{\frac{1}{2}}$$

Therefore, if $\alpha \le \theta \le \beta$, then $L = \int_{\theta=\alpha}^{\beta}\left(r^2 + (r')^2\right)^{\frac{1}{2}}d\theta$.

III. COMPUTATION OF THE AREA AND THE ARC LENGTH IN THE POLAR COORDINATE

The area in a curve

We have a graph of equation $r = 4\cos 2\theta$. The graph as Figure 1. We want to find that area. First, we must determine the rotation angle. We can choose $r = 0$ and $r = 4$. If $r = 0$, then the rotation angle is $\theta = \frac{\pi}{4}$. If $r = 4$, then the rotation angle is $\theta = 0$. Therefore, we can take the rotation angle is $0 \le \theta \le \frac{\pi}{4}$.

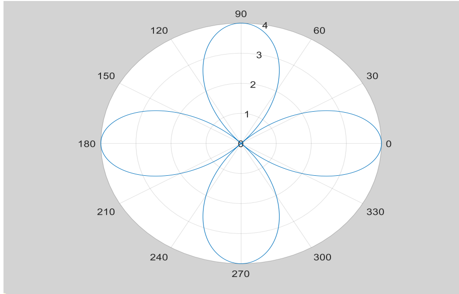

Figure 1. The graph $r = 4\cos 2\theta$.

Based on Equation 2, the area is

$$A_1 = 4\int_{\theta=0}^{\frac{\pi}{4}}\left[4\cos 2\theta\right]^2 d\theta = 32\int_{\theta=0}^{\frac{\pi}{4}}\left[1 + \cos 4\theta\right]d\theta$$
$$= 32\left(\theta + \tfrac{1}{4}\sin 4\theta\right|_{\theta=0}^{\frac{\pi}{4}} = 8\pi$$

The area in two curves

Let there are two curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$. The illustration of the two curves is as shown in Figure 2.

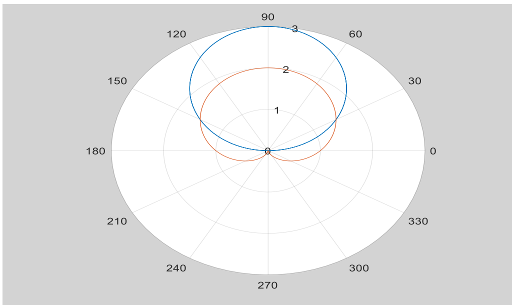

Figure 2. The graph $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

This will be looked for five regional areas, namely

a. The area of the region inside $r = 1 + \sin\theta$.

b. The area of the region inside $r = 3\sin\theta$, but outside $r = 1 + \sin\theta$.

c. The area of the region inside both $r = 3\sin\theta$ and $r = 1 + \sin\theta$.

d. The area of the entire region $r = 3$.

e. The area of the region outside both $r = 3\sin\theta$ and $r = 1 + \sin\theta$.

a. The area of the region inside $r = 1 + \sin\theta$.

We choose $r = 0$ and $r = 2$. If $r = 0$, then the rotation angle $\theta = -\frac{\pi}{2}$. If $r = 2$, then the rotation angle $\theta = \frac{\pi}{2}$.

Therefore, we can choose the rotation angle is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. The area is

$$A_1 = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \; d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{3}{2} + 2\sin\theta - \tfrac{1}{2}\cos 2\theta\right] d\theta$$

$$= (\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} = \tfrac{3}{2}\left[\tfrac{\pi}{2} + \tfrac{\pi}{2}\right] = 1\tfrac{1}{2}\pi$$

b. The area of the region inside $r = 3\sin\theta$, but outside $r = 1+\sin\theta$.

First, we look for the intersection point of the two curves, by

$r = r \;\Leftrightarrow\; 1+\sin\theta = 3\sin\theta \;\Leftrightarrow\; \sin\theta = \frac{1}{2} \;\Leftrightarrow\; \theta = \frac{1}{6}\pi \vee \frac{5}{6}\pi$. Therefore, the rotation angle is $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$. The intersection points are $A\left(\frac{3}{2}, \frac{1}{6}\pi\right)$ and $B\left(\frac{3}{2}, \frac{5}{6}\pi\right)$. The area of this region is

$$A_2 = \tfrac{1}{2}\int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

$$= \tfrac{1}{2}\int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[8\sin^2\theta - 1 - 2\sin\theta\right] d\theta$$

$$= \tfrac{1}{2}\int_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[3 - 4\cos 2\theta - 2\sin\theta\right] d\theta$$

$$= \tfrac{1}{2}(3\theta - 2\sin 2\theta + 2\cos\theta\Big|_{\theta=\frac{\pi}{6}}^{\frac{5}{6}\pi} = \tfrac{1}{2}\left[3\left(\tfrac{4}{6}\pi\right)\right] = \pi$$

c. The area of the region inside both $r = 3\sin\theta$ and $r = 1+\sin\theta$.

We could choose the rotation angle is $\frac{5\pi}{6}$ to $\frac{\pi}{6}$. The area is

$$A_3 = \tfrac{1}{2}\int_{\theta=\frac{5\pi}{6}}^{\frac{\pi}{6}} \left[(1+\sin\theta)^2 - (3\sin\theta)^2\right] d\theta$$

$$= \tfrac{1}{2}\int_{\theta=\frac{5\pi}{6}}^{\frac{\pi}{6}} \left[1 + 2\sin\theta - 8\sin^2\theta\right] d\theta$$

$$= \tfrac{1}{2}\int_{\theta=\frac{5\pi}{6}}^{\frac{\pi}{6}} \left[2\sin\theta + 4\cos 2\theta - 3\right] d\theta$$

$$= \tfrac{1}{2}(-2\cos\theta + 2\sin 2\theta - 3\theta\Big|_{\theta=\frac{5\pi}{6}}^{\frac{\pi}{6}}$$

$$= \tfrac{1}{2}\left[3\left(\tfrac{4}{6}\pi\right)\right] = \pi$$

d. The area of the entire region $r = 3$.

We choose the rotation angle is $0 \le \theta \le 2\pi$. Then the area is

$$A_3 = \tfrac{1}{2}\int_{\theta=0}^{2\pi} 3^2 \; d\theta = \tfrac{1}{2}(9\theta\Big|_{\theta=0}^{2\pi} = \tfrac{1}{2}\left[9(2\pi)\right] = 9\pi$$

e. The area of the region outside both $r = 3\sin\theta$ and $r = 1+\sin\theta$.

There are three areas that are interrelated in this case. The areas are A_1, A_2 and A_4. Its area is $A_5 = A_4 - A_1 - A_2 = 6\frac{1}{2}\pi$.

The Arc Length of a Curve

If we choose the curve $r = \cos\theta$. It has a picture as Figure 3.

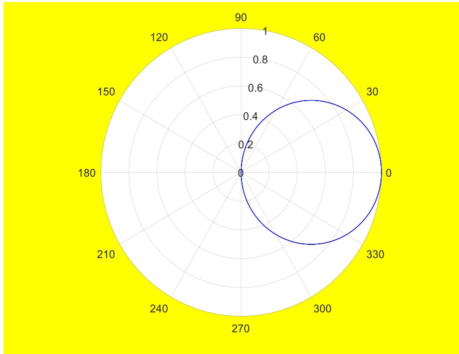

Figure 3. The graph $r = \cos\theta$

We will choose $r = 0$ and $r = 1$. If $r = 0$, then the rotation angle is $\theta = \frac{\pi}{2}$. If $r = 1$, then the rotation angle is $\theta = 0$. Therefore, we can choose the rotation angle is $0 \le \theta \le \frac{\pi}{2}$. Based on Equation 4, the arc length is

$$L = 2\int_{\theta=0}^{\frac{\pi}{2}} \left(r^2 + (r')^2\right)^{\frac{1}{2}} d\theta$$

$$= 2\int_{\theta=0}^{\frac{\pi}{2}} \left(\cos^2\theta + (\sin^2\theta)\right)^{\frac{1}{2}} d\theta = 2\theta\Big|_{\theta=0}^{\frac{\pi}{2}} = \pi$$

Theorem 3.

If there is a curve $r = a\cos\theta$, where $a > 0$, $0 \le \theta \le \pi$. The arc length of the curve is $a\pi$.

Proof:

Let $r = a\cos\theta$ so $r' = -a\sin\theta$. Based on Equation 4, the arc length of this curve is

$$L = \int_{\theta=0}^{\pi} \left((a\cos\theta)^2 + (-a\sin\theta)^2\right)^{\frac{1}{2}} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(a^2\cos^2\theta + a^2\sin^2\theta\right)^{\frac{1}{2}} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(a^2\right)^{\frac{1}{2}} d\theta = a\theta\Big|_{\theta=0}^{\pi} = a\pi$$

If we do a simulation with several constants as shown in Table 1.

Table 1. Simulation value of curve $r = a\cos\theta$

a	$r = a\cos\theta$	L
1	$r = \cos\theta$	π
5	$r = 5\cos\theta$	5π
10	$r = 10\cos\theta$	10π
15	$r = 15\cos\theta$	15π
20	$r = 20\cos\theta$	20π

If we choose the curve $r = \sin\theta$. It has a picture as Figure 4.

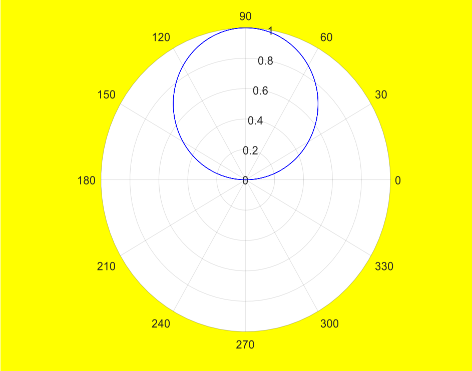

Figure 4. The graph $r = \sin\theta$

We will choose $r=0$ and $r=1$. If $r=0$, then the rotation angle is $\theta=0$. If $r=1$, the rotation angle is $\theta=\frac{\pi}{2}$. Therefore, we can choose the rotation angle is $0\le\theta\le\frac{\pi}{2}$. Based on Equation 4, the arc length is

$$L = 2\int_{\theta=0}^{\frac{\pi}{2}}\left(r^2+(r')^2\right)^{\frac{1}{2}}d\theta = 2\int_{\theta=0}^{\frac{\pi}{2}}\left((\sin^2\theta)+\cos^2\theta\right)^{\frac{1}{2}}d\theta$$

$$= 2\theta\Big|_{\theta=0}^{\frac{\pi}{2}} = \pi$$

Theorem 4.

If there is a curve $r=a\sin\theta$, where $a>0$, $0\le\theta\le\pi$. Based on Equation 4, the arc length of the curve is $a\pi$.

Proof:

Let $r=a\sin\theta$, and $r'=a\cos\theta$. The arc length of this curve is

$$L = \int_{\theta=0}^{\pi}\left((a\sin\theta)^2+(a\cos\theta)^2\right)^{\frac{1}{2}}d\theta$$

$$= \int_{\theta=0}^{\pi}\left(a^2\sin^2\theta+a^2\cos^2\theta\right)^{\frac{1}{2}}d\theta$$

$$= \int_{\theta=0}^{\pi}\left(a^2\right)^{\frac{1}{2}}d\theta = a\theta\Big|_{\theta=0}^{\pi} = a\pi$$

Table 2. Simulation value of curve $r=a\sin\theta$

a	$r=a\sin\theta$	L
1	$r=\sin\theta$	π
5	$r=5\sin\theta$	5π
10	$r=10\sin\theta$	10π
15	$r=15\sin\theta$	15π
20	$r=20\sin\theta$	20π

If we have a curve $r=1+\cos\theta$. We will find the area and the arc length of this curve. The graph of the curve $r=1+\cos\theta$ is presented as in Figure 5.

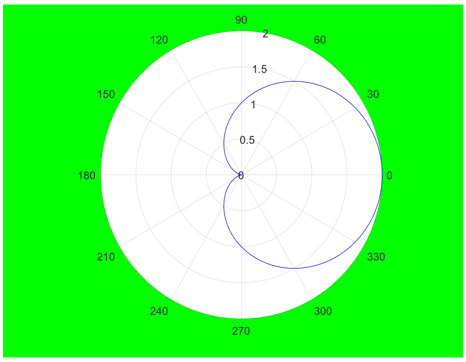

Figure 5. The graph $r=1+\cos\theta$

The area of the graph $r=1+\cos\theta$ be described below. We choose $r=0$ and $r=2$. If $r=0$, then the rotation angle is $\theta=\pi$. If $r=2$, the rotation angle be obtained $\theta=0$. Therefore, we can make the rotation angle is $0\le\theta\le\pi$. Then based on Equation 2, the region area can be calculated by

$$A = \int_{\theta=0}^{\pi}(1+\cos\theta)^2\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos 2\theta\right]d\theta$$

$$= (\tfrac{3}{2}\theta+2\sin\theta+\tfrac{1}{4}\sin 2\theta\Big|_{\theta=0}^{\pi} = \left[\tfrac{3}{2}(\pi)\right] = \tfrac{3}{2}\pi$$

Theorem 5.

If there is a curve $r=a+a\cos\theta$, where $a>0$, $0\le\theta\le\pi$. Then the region area of the graph is $\frac{3}{2}\pi a^2$.

Proof:

Let $r=a+a\cos\theta$, then based on Equation 2 the region area is

$$A = \int_{\theta=0}^{\pi}(a+a\cos\theta)^2\,d\theta$$

$$= a^2\int_{\theta=0}^{\pi}\left[\tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos 2\theta\right]d\theta$$

$$= a^2\,(\tfrac{3}{2}\theta+2\sin\theta+\tfrac{1}{4}\sin 2\theta\Big|_{\theta=0}^{\pi}$$

$$= a^2\left[\tfrac{3}{2}(\pi)\right] = a^2\,\tfrac{3}{2}\pi = \tfrac{3}{2}\pi a^2$$

While the arc length of the curve $r=1+\cos\theta$ with $r'=-\sin\theta$ be computed by Equation 4, as below

$$L = 2\int_{\theta=0}^{\pi}\left[r^2+(r')^2\right]^{\frac{1}{2}}d\theta = 2\int_{\theta=0}^{\pi}(2+2\cos\theta)^{\frac{1}{2}}d\theta$$

$$= 4\int_{\theta=0}^{\pi}\cos\tfrac{\theta}{2}\,d\theta = 4(2\sin\tfrac{\theta}{2}\Big|_{\theta=0}^{\pi} = 8$$

Theorem 6

Let there is a curve $r=a+a\cos\theta$, where $a>0$, $0\le\theta\le\pi$. Then the arc length of the curve $r=a+a\cos\theta$ is $8a$.

Proof:

Let $r = a + a\cos\theta$, so $r' = -a\sin\theta$. Then based on Equation 4, calculation of the arc length be obtained result

$$L = 2\int_{\theta=0}^{\pi}\left[r^2 + (r')^2\right]^{\frac{1}{2}} d\theta = 2a\int_{\theta=0}^{\pi}(2+2\cos\theta)^{\frac{1}{2}} d\theta$$
$$= 2\sqrt{2}\,a\int_{\theta=0}^{\pi}(1+\cos\theta)^{\frac{1}{2}} d\theta = 4a\int_{\theta=0}^{\pi}\cos\tfrac{\theta}{2}\, d\theta$$
$$= 4a(2\sin\tfrac{\theta}{2}\Big|_{\theta=0}^{\pi} = 8a$$

If we do a simulation the curve $r = a + a\cos\theta$ with several constants about the area and the arc length curve as shown in Table 3.

Table 3. Simulation value of curve $r = a + a\cos\theta$

a	$r = a + a\cos\theta$	A	L
1	$r = 1+\cos\theta$	$1\frac{1}{2}\pi$	8
2	$r = 2+2\cos\theta$	6π	16
3	$r = 3+3\cos\theta$	$13\frac{1}{2}\pi$	24
4	$r = 4+4\cos\theta$	24π	32
5	$r = 5+5\cos\theta$	$37\frac{1}{2}\pi$	40

IV. CONCLUSION

Polar coordinate is two-dimensional coordinate system that defines at position of a point on a plane based on the distance *r* from the origin and the angle θ with respect to the positive *x*-axis. The region area is a quantity that measures two-dimensional shape bounded by a curve or line. The arc length is the length of a curved line connecting two points on the circumference of the curve formed. The area of a region formed by a curve is directly proportional to the square of the curve function, while the arc length on the curve is influenced by the square of the curve function and its first derivative. For function that is constant multiplication with sine and cosine functions, the arc length is a multiple of the constant. For function that is constant multiplication with one plus the cosine function of the angle, the area of the region is a multiple of the square of the constant, while the arc length is a multiple of eight.

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