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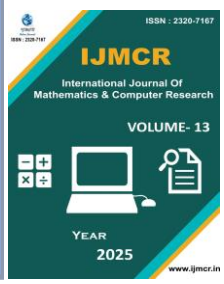

Modelling the Nonlinear Incidence Rate of Drug Addiction: A Mathematical Approach

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ABSTRACT

Drug addiction remains a critical public health issue impacting individuals and communities worldwide. This study presents a mathematical model designed to analyze the dynamics of drug addiction transmission within a population. The model features nonlinear incidence rates and categorizes the population into five compartments: susceptible, exposed, addicted, quarantined or rehabilitated, and recovered individuals. Key parameters influencing transmission and recovery processes are identified and examined, particularly intervention factors that reduce addiction spread. The basic reproduction number, $\Re_0$, serves as a threshold determining the stability of equilibrium states. When $\Re_0 < 1$, the drug-free equilibrium is stable, whereas $\Re_0 > 1$ indicates the persistence of addiction within the population. A forward bifurcation is observed at $\Re_0 = 1$, signifying a critical transition point in the system's dynamics where stability shifts between equilibria. The analysis highlights that preventive and rehabilitative interventions significantly influence the control of addiction prevalence. These findings offer valuable insights for designing effective public health strategies aimed at mitigating the drug addiction epidemic.

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KEYWORDS: Mathematical Modelling; Forward Bifurcation; Stability Analysis; Basic Reproduction Number; Drug Abuse

I. INTRODUCTION

Drug addiction is a serious problem that affects all layers of society worldwide. The impacts of drug addiction greatly influence individual well-being, both physically and mentally. These consequences also extend broadly to social, economic, and public health levels. Drug use issues have been reported in more than 140 countries, affecting their economies and development [1]. These effects encompass various diseases and social challenges. Recent data estimate the global number of people who inject drugs in 2021 to be 13.2 million, which is 18 percent higher than previous estimates. Globally, more than 296 million people used drugs in 2021, an increase of 23 percent over the previous decade. Meanwhile, the number of people suffering from drug use disorders has skyrocketed to 39.5 million, rising by 45 percent over ten years [2]. International law prohibits the non-medical use of illegal drugs. In recent years, drug abuse has become a critical issue because it destroys the future of young generations, who are the hope of developing countries [1].

Mathematical modeling and simulation are powerful and effective tools for solving a wide variety of problems because they enable the representation, analysis, and experimentation of complex systems in a virtual environment. Many instances of disease spread have been effectively addressed through mathematical modeling, which enables researchers to anticipate transmission patterns, evaluate intervention strategies, and develop optimal control measures—all without resorting to costly or hazardous direct experimentation. By transforming real-world phenomena into equations and logical relationships, mathematical models can be analyzed through simulations to provide deep insights that assist decision-making in public health, epidemiology, and other sectors. These approaches have been extensively documented and can be referenced in studies [3], [4], [5], [6], [7], [8], and [9] for further reading and validation.

Several studies on mathematical modeling of drug abuse have already been conducted. Study [10] concludes that drug abuse significantly threatens individual health and socio-economic order at the national level. Social programs through educational campaigns in learning institutions, social media, and community organizations must be intensified to raise public awareness about the dangers of drug abuse. Study [11] shows that media effects on human populations do not change equilibrium stability but can influence the number of drug addicts and analyzes sensitivity to find effective control measures for treatment. Study [12] demonstrates that family and community health education can affect drug transmission spread, with a combination of both being more effective in reducing drug prevalence. Study [13] explains that anti-drug education and media coverage play a major role, so prevention and treatment should work simultaneously to control the spread of drug abuse more cost-effectively and quickly. Study [14] shows that appropriate policies and rehabilitation programs and their implementation will reduce drug infection transmission in Taraba North State, Nigeria.

In this study, the research from [14] is modified by using a nonlinear incidence rate and considering a quarantine class for drug addicts. The analysis results are expected to provide an overview that using a nonlinear incidence rate and quarantine class can suppress individuals from abusing drugs.

II. METHOD AND RESULT

A. Model Formulation

The mathematical model for drug addiction with a nonlinear incidence rate is structured into five distinct populations, each representing a different stage or status in relation to drug addiction. These populations are denoted as *S(t), E(t), A(t), Q(t),* and *R(t)*, each capturing the dynamics of individuals as they transition through various phases of addiction and recovery. *S(t)* represents individuals who are susceptible to addiction if they interact with addicted individuals, which occurs through social interaction and environmental influence; *E(t)* represents individuals exposed to drugs, where the susceptible individuals have been affected by addicted individuals but have not yet become fully addicted; *A(t)* represents individuals who are addicted to illegal drugs and can influence susceptible individuals; *Q(t)* represents addicted individuals who are trying to quit or are in recovery; and *R(t)* represents individuals who have recovered from drug addiction. Naturally, the value of each variable must be positive (or zero) over time given the initial conditions.

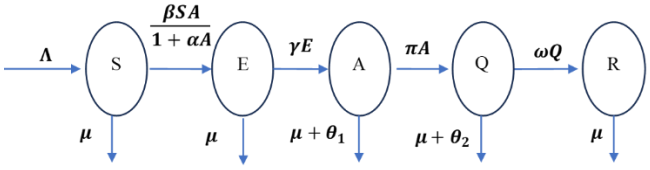

Figure 1. Compartmental Diagram of the Spread of Illegal Drug Abuse

The mathematical model of drug addiction with a nonlinear incidence rate can be represented by the following system of ordinary differential equations.

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta SA}{1+\alpha A} - \mu S \\
\frac{dE}{dt} &= \frac{\beta SA}{1+\alpha A} - (\gamma+\mu)E \\
\frac{dA}{dt} &= \gamma E - (\mu+\theta_1+\pi)A \\
\frac{dQ}{dt} &= \pi A - (\mu+\theta_2+\omega)Q \\
\frac{dR}{dt} &= \omega Q - \mu R
\end{aligned} \tag{1}$$

With $S(t_0) = S_0 \geq 0, E(t_0) = E_0 \geq 0, A(t_0) = A_0 \geq 0,$ $Q(t_0) = Q_0 \geq 0$, and $R(t_0) = R_0 \geq 0$ (2)

The function $f(\alpha, A) = \beta SA / (1+\alpha A)$ expresses the decrease in the rate of drug spreading as the number of addicted individuals increases due to an inhibitory effect parameter α. This parameter can represent the presence of interventions that reduce the effectiveness of drug abuse. For example, public health interventions such as prevention programs or education about the dangers of illegal drugs can increase the value of α, which, if effective, will reduce the drug abuse rate. The descriptions of the parameters used can be found in detail in TABLE I.

TABLE I. DESCRIPTION AND VALUES OF PARAMETERS

Parameter	Description	Value	Source
Λ	Human recruitment	1-30	[15]
β	Rate of drug influence	0.005	[15]
μ	Natural death rate	0.031	[15]
θ_1	Death rate of individuals already addicted to drugs	0.01	[15]
θ_2	Death rate of individuals undergoing rehabilitation	0.01	[15]
π	Rate of movement from compartment A to Q	0.044	[15]
γ	Rate of movement from compartment E to A	0.024	[15]
ω	Rate of movement from compartment Q to R	0.01	[15]

B. Positive Solutions and The Boundedness of The Solution

Given the initial values at $S(t_0) = S_0 \geq 0, E(t_0) = E_0 \geq 0, A(t_0) = A_0 \geq 0, Q(t_0) = Q_0 \geq 0,$, and $R(t_0) = R_0 \geq 0$. Observe that:

$$\frac{dS}{dt} = \Lambda - \frac{\beta SA}{1+\alpha A} - \mu S \geq -\left(\frac{\beta A}{1+\alpha A} + \mu\right) S$$

Integrate both sides:

$$\int_{S(0)}^{S(t)} \frac{dS}{S} \geq \int_0^t -\left(\frac{\beta A}{1+\alpha A}+\mu\right) dt$$

$$\ln|S(t)| - \ln|S(0)| \geq -\left(\frac{\beta A}{1+\alpha A}+\mu\right)(t)$$

$$\ln\left|\frac{S(t)}{S(0)}\right| \geq -\left(\frac{\beta A}{1+\alpha A}+\mu\right)(t)$$

$$\left|\frac{S(t)}{S(0)}\right| \geq e^{-\left(\frac{\beta A}{1+\alpha A}+\mu\right)(t)}$$

$$S(t) \geq S(0)e^{-\left(\frac{\beta A}{1+\alpha A}+\mu\right)(t)} \geq 0$$

For the compartment E, A, and R:

$$\frac{dE}{dt} = \frac{\beta SA}{1+\alpha A} - (\gamma+\mu)E$$
$$\geq -(\gamma+\mu)E$$
$$\Leftrightarrow \frac{dE}{E} \geq -(\gamma+\mu)dt$$
$$\Leftrightarrow \int_{E(0)}^{E(t)} \frac{1}{E} dE \geq \int_0^t -(\gamma+\mu)\, dt$$
$$\Leftrightarrow E(t) \geq E(0)e^{-(\gamma+\mu)t} \geq 0$$

$$\frac{dA}{dt} = \gamma E - (\mu+\theta_1+\pi)A$$
$$\geq -(\mu+\theta_1+\pi)A$$
$$\Leftrightarrow \int_{A(0)}^{A(t)} \frac{1}{A} dA \geq \int_0^t -(\mu+\theta_1+\pi)\, dt$$
$$\Leftrightarrow A(t) \geq A(0)e^{-(\mu+\theta_1+\pi)} \geq 0$$

$$\frac{dQ}{dt} = \pi A - (\mu+\theta_2+\omega)Q$$
$$\geq -(\mu+\theta_2+\omega)Q$$
$$\Leftrightarrow \int_{Q(0)}^{Q(t)} \frac{1}{Q}\, dQ \geq \int_0^t -(\mu+\theta_2+\omega)Q$$
$$Q(t) \geq Q(0)e^{(\mu+\theta_2+\omega)t} \geq 0$$

$$\frac{dR}{dt} = \omega A - \mu R \geq \mu R$$
$$\Leftrightarrow \int_{R(0)}^{R(t)} \frac{1}{R} dR \geq \int_0^{t-} \mu R\, dt$$
$$\Leftrightarrow R(t) \geq R(0)e^{-\mu t} \geq 0$$

It can be concluded that the populations in each compartment satisfy:

$S(t) \geq 0, E(t) \geq 0, A(t) \geq 0, Q(t) \geq 0,\ R(t) \geq 0 \quad \forall t > 0$

The solution of system (1) with initial condition is bounded within the set $\Omega = \left\{(S,E,A,Q,R)\, \epsilon\, \mathbb{R}^5 : 0 \leq N \leq \frac{\Lambda}{\mu}\right\}$.

Proof:

By adding all the differential equations given for $S(t)$, $E(t)$, $A(t)$, $Q(t)$, and $R(t)$, we write an equation for the total population $N(t)$ as follows:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dA}{dt} + \frac{dQ}{dt} + \frac{dR}{dt}$$
$$= \Lambda - \mu(S+E+A+Q+R) - (\theta E + \theta_1 A + \theta_2 Q)$$
$$= \Lambda - \mu N - (\theta_1 A + \theta_2 Q)$$
$$\leq \Lambda - \mu N$$

After simplification, this differential inequality can be solved linearly. The solution describes how the total population changes according to the parameters.

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

Noting that

$$\frac{dN}{dt} + \mu N = \Lambda$$

using the integrating factor $u(t) = e^{\int \mu dt} = e^{\mu t}$, we have:

$$\left(\frac{dN}{dt} + \mu N\right) e^{\mu t} = \Lambda e^{\mu t}$$
$$\Leftrightarrow \frac{d}{dt}(N(t)e^{\mu t}) = \Lambda e^{\mu t}$$

Integrating both sides yields:

$$N(t)e^{\mu t} = \frac{\Lambda}{\mu} e^{\mu t} + C$$

At $t = 0$, $N(0) - \frac{\Lambda}{\mu} = C$.

Hence, $N(t) = \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}$, Applying Gronwall's inequality [16], we have:

$N(t) \leq \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}$ and as $t \to \infty$, $N(t) \leq \frac{\Lambda}{\mu}$

Thus, all solutions remain within the set. If $N(0) < \frac{\Lambda}{\mu}$, then as $t \to \infty$, $N(t) \to \frac{\Lambda}{\mu}$, meaning $0 \leq N \leq \frac{\Lambda}{\mu}$.

C. Existence and Uniqueness of Solutions

Suppose the function $\boldsymbol{f}$ in System (1) satisfies the Lipschitz condition in the set $\Omega \subset \mathbb{R}_+^5$ if there exists a constant $k > 0$ such that

$$|f(X_1) - f(X_2)| < k|X_1 - X_2|, \forall X_1, X_2 \in \Omega$$

The solution of System (1) with initial conditions $S(0), E(0), A(0), Q(0), R(0)$ denoted by $S(t), E(t), A(t), Q(t), R(t)$ exists and is unique in $\Omega \subset \mathbb{R}_+^5$. According to the Grossman and Derrick theorem [17], System (1) satisfies the Lipschitz's condition if the partial derivatives $\frac{\partial f_i}{\partial X_j}$ are continuous and $\frac{\partial f_i}{\partial X_j} < \infty, \forall i, j = 1,2,3,4,5$.

Proof:

Consider system (1), and let

$$X = (S(t), E(t), A(t), Q(t), R(t))$$

The components of $\boldsymbol{f}$ are given as:

$$f_1 = \frac{dX_1}{dt} = \Lambda - \frac{\beta X_1 X_3}{1+\alpha X_3} - \mu X_1$$
$$f_2 = \frac{dX_2}{dt} = \frac{\beta X_1 X_3}{1+\alpha X_3} - (\gamma+\mu)X_2$$
$$f_3 = \frac{dX_3}{dt} = \gamma X_2 - (\mu+\theta_1+\pi)X_3$$
$$f_4 = \frac{dX_4}{dt} = \pi X_3 - (\mu+\theta_2+\omega)X_4$$
$$f_5 = \frac{dR}{dt} = \omega X_4 - \mu X_5$$

The partial derivatives are bounded and continuous as follows:

$$\left|\frac{\partial f_1}{X_1}\right| = \left|-\frac{\beta A}{A\alpha+1}-\mu\right| < K < \infty, \left|\frac{\partial f_1}{X_2}\right| = |0| < K < \infty, \left|\frac{\partial f_1}{X_3}\right| = \left|-\frac{\beta S}{A\alpha+1}+\frac{\beta SA\alpha}{(A\alpha+1)^2}\right| < K < \infty, \left|\frac{\partial f_1}{X_4}\right| = |0| < K < \infty, \left|\frac{\partial f_1}{X_5}\right| = |0| < K < \infty$$

$$\left|\frac{\partial f_2}{X_1}\right| = \left|\frac{\beta A}{A\alpha+1}\right| < K < \infty, \left|\frac{\partial f_2}{X_2}\right| = |\gamma+\mu| < K < \infty, \left|\frac{\partial f_2}{X_3}\right| = \left|\frac{\beta S}{A\alpha+1}-\frac{\beta SA\alpha}{(A\alpha+1)^2}\right| < K < \infty, \left|\frac{\partial f_2}{X_4}\right| = |0| < K < \infty, \left|\frac{\partial f_2}{X_5}\right| = |0| < K < \infty$$

$$\left|\frac{\partial f_3}{X_1}\right| = |0| < K < \infty, \left|\frac{\partial f_3}{X_2}\right| = |\gamma| < K < \infty, \left|\frac{\partial f_3}{X_3}\right| = |\mu+\theta_1+\pi| < K < \infty, \left|\frac{\partial f_3}{X_4}\right| = |0| < K < \infty, \left|\frac{\partial f_3}{X_5}\right| = |0| < K < \infty$$

$$\left|\frac{\partial f_4}{X_1}\right| = |0| < K < \infty, \left|\frac{\partial f_4}{X_2}\right| = |0| < K < \infty, \left|\frac{\partial f_4}{X_3}\right| = |\pi| < K < \infty, \left|\frac{\partial f_4}{X_4}\right| = |\mu+\theta_2+\omega| < K < \infty, \left|\frac{\partial f_4}{X_5}\right| = |0| < K < \infty$$

$$\left|\frac{\partial f_5}{X_1}\right| = |0| < K < \infty, \left|\frac{\partial f_5}{X_2}\right| = |0| < K < \infty, \left|\frac{\partial f_5}{X_3}\right| = |0| < K < \infty, \left|\frac{\partial f_5}{X_4}\right| = |\omega| < K < \infty, \left|\frac{\partial f_5}{X_5}\right| = |\mu| < K < \infty$$

Since the partial derivatives $\partial \boldsymbol{f_i} / \partial \boldsymbol{X_j}, \forall i,j = 1,2,3,4,5$ are continuous functions with bounded values, System (1) satisfies the Lipschitz condition. Therefore, the solution of System (1) with initial conditions $S(0), E(0), A(0), Q(0), R(0)$ exists and is unique in $\Omega \subset \mathbb{R}^5_+$.

D. Basic Reproduction Number and Equilibrium Points

The epidemic threshold value plays an important role in estimating the number of secondary infections that occur due to the initial infection in a population. The basic reproduction number, or threshold value, can be determined using the Next Generation Matrix [18]. In the absence of drug-addicted individuals, $E = 0$ and $A = 0$, an addiction-free equilibrium point is obtained, which will be used to determine the basic reproduction number. System (1) can be written as follows:

$$\dot{\boldsymbol{x}} = F - V = \begin{pmatrix} \frac{\beta SA}{1+A\alpha} \\ 0 \end{pmatrix} - \begin{pmatrix} (\gamma+\mu)E \\ (\mu+\theta_1+\pi)A - \gamma E \end{pmatrix}.$$

The infected compartments and addicted individuals in the system are E and A. Next, the Jacobian matrices $\mathbb{F}$ and $\mathbb{V}$ at the addiction-free equilibrium $\boldsymbol{E_0}$ are found as:

$$\mathbb{F} = \mathbb{J}_{F|E_0} = \begin{pmatrix} 0 & \frac{\beta\Lambda}{\mu} \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{V} = \mathbb{J}_{V|E_0} = \begin{pmatrix} \gamma+\mu & 0 \\ -\gamma & \mu+\theta_1+\pi \end{pmatrix}$$

$$\mathbb{F}\mathbb{V}^{-1} = \begin{pmatrix} \frac{\beta\Lambda\gamma}{\mu(\gamma+\mu)(\mu+\theta_1+\pi)} & \frac{\beta\Lambda}{\mu(\mu+\theta_1+\pi)} \\ 0 & 0 \end{pmatrix}$$

Therefore, the dominant eigenvalue $\Re_0$ of the matrix $\mathbb{F}\mathbb{V}^{-1}$ is given by

$$\Re_0 = \frac{\beta\Lambda\gamma}{\mu(\gamma+\mu)(\mu+\theta_1+\pi)}.$$

Theorem 1. Given $\Re_0 = \frac{\beta\Lambda\gamma}{\mu(\gamma+\mu)(\mu+\theta_1+\pi)}$,

1. If $\Re_0 \leq 1$, then there exists a drug addiction-free equilibrium point $E_1 = \left(\frac{\Lambda}{\mu}, 0,0,0,0\right)$.
2. If $\Re_0 > 1$, then there exists a drug addiction equilibrium point $E_2 = (S_2^*, E_2^*, A_2^*, Q_2^*, R_2^*)$.

Proof:

All equilibrium points are determined by setting each time derivative equal to zero:

$$\frac{dS}{dt} = \frac{dE}{dt} = \frac{dA}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0$$

We solve this system of algebraic equations by substitution between the equations. Starting with the third equation, we find the relation between two variables:

$$0 = \gamma E - (\mu + \theta_1 + \pi)A$$
$$\Leftrightarrow A = \frac{\gamma E}{\mu+\theta_1+\pi}$$

Substitute A into the first, second, and fourth equation, the solutions for the equilibrium points can be found by solving these equations simultaneously. The equilibrium points found are:

1. The drug addiction-free equilibrium point $E_1 = \left(\frac{\Delta}{\mu}, 0,0,0,0\right)$
2. The drug addiction equilibrium point: $E_2 = (S_2^*, E_2^*, A_2^*, Q_2^*, R_2^*)$

with

$$S_2^* = \frac{\Lambda(1+A\alpha)}{(\alpha\mu+\beta)A+\mu}$$
$$E_2^* = \frac{(\mu+\theta_1+\pi)(\Re_0-1)\mu}{(\alpha\mu+\beta)\gamma}$$
$$A_2^* = \frac{\gamma E}{\mu+\theta_1+\pi}$$
$$Q_2^* = \frac{\pi\gamma E}{(\mu+\theta_2+\omega)(\mu+\theta_1+\pi)}$$
$$R_2^* = \frac{\omega\pi\gamma E}{\mu(\mu+\theta_2+\omega)(\mu+\theta_1+\pi)}$$

It can be noted that if $\Re_0 \leq 1$, then $E_1 = (\Delta/\mu, 0,0,0,0)$ and if $\Re_0 > 1$, then E_2 exists as described above.

E. Stability of Equilibrium Points and Bifurcation

Theorem 2.

1. The drug addiction-free equilibrium point of System (1) is locally asymptotically stable if $\Re_0 < 1$, and unstable if $\Re_0 > 1$.
2. The drug addiction-free equilibrium point of System (1) is globally stable in $\Omega \subset \mathbb{R}^5_+$ if $\Re_0 < 1$

3. The drug addiction equilibrium point of System (1) is locally asymptotically stable if $\Re_0 > 1$.

Proof:

Evaluated at the disease-free equilibrium E_1, the Jacobian matrix becomes:

$$J_{E|E_0} = \begin{pmatrix} -\mu & 0 & -\frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & -k_1 & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & \gamma & -k_2 & 0 & 0 \\ 0 & 0 & \pi & -k_3 & 0 \\ 0 & 0 & 0 & \omega & -\mu \end{pmatrix}$$

The characteristic equation is:

$$\frac{\left((\Lambda\beta\gamma - k_1k_2\mu - k_1\lambda\mu - k_2\lambda\mu - \lambda^2\mu)(\mu + \lambda)^2((k_3 + \lambda)\right)}{-\mu} = 0$$

From this, we identify eigenvalues: $\lambda_1 = -k_3, \lambda_{2,3} = -\mu$. Since all parameters are positive, we have $\lambda_1, \lambda_2\, \lambda_3 < 0$. The remaining eigenvalues are the roots of $(\Lambda\beta\gamma - k_1k_2\mu - (k_1 + k_2)\mu\lambda - \lambda^2\mu) = 0$ Because $\Lambda\beta\gamma = \Re_0 k_1k_2\mu$, this quadratic equation can be rewritten as: $\lambda^2\mu + (k_1 + k_2)\mu\lambda - (\Re_0 - 1)k_1k_2\mu = 0$ The Hurwitz matrix for this quadratic polynomial is:

$$\begin{bmatrix} a_1 & a_0 \\ 0 & a_2 \end{bmatrix}$$

Where $a_0 = \mu$, $a_1 = (k_1 + k_2)\mu$ and $a_2 = -(\Re_0 - 1)k_1k_2\mu$

Next, the Hurwitz determinants are: $\Delta_1 = |a_1| = \mu > 0$, $\Delta_2 = a_1a_2 = \mu(-(\Re_0 - 1)k_1k_2\mu)$. If $\Re_0 < 1$, then $\Delta_2 > 0$. According to the Routh-Hurwitz criteria, λ_4 and λ_5 are negative. Therefore, all roots of the characteristic equation are negative, and the equilibrium point E_1 is locally asymptotically stable. Conversely, if $\Re_0 > 1$, E_1 is not stable. The drug addiction equilibrium point of System (1) is locally asymptotically stable if $\Re_0 > 1$ ■

Now, we discuss about global stability of the drug addiction-free equilibrium point. Define the Lyapunov function:

$$V = \frac{\gamma E}{(\gamma + \mu)(\mu + \theta_1 + \pi)} + \frac{A}{\mu + \theta_1 + \pi}$$

This function is non-negative and zero only the drug addiction-free equilibrium point $E = 0, A = 0$. The time derivative of V along the trajectories of the system 1:

$$\frac{dV}{dt} = \frac{\gamma\beta SA}{(\gamma + \mu)(\mu + \theta_1 + \pi)(1 + \alpha A)} - A$$

Since $S \le S_1 = \frac{\Lambda}{\mu}$ and $\frac{1}{1+\alpha A} \le 1$, we have

$$\frac{dV}{dt} \le \frac{\gamma\beta S_1 A}{(\gamma + \mu)(\mu + \theta_1 + \pi)} - A = (\Re_0 - 1)A$$

If $\Re_0 \le 1$, then $\frac{dV}{dt} \le 0$ with equality only when $A = 0$. By Lassalle's Invariance Principle, the largest invariant set where $\frac{dV}{dt} = 0$ is $\Omega \subset \mathbb{R}^5_+$. Therefore, E_1 is globally asymptotically stable when $\Re_0 \le 1$ ■

We investigate the stability of E_2, the Jacobian matrix of the system evaluated at the equilibrium point E_2 is given by:

$$\begin{pmatrix} -\frac{\beta A_1}{A_1\alpha + 1} - \mu & 0 & c_1 & 0 & 0 \\ \frac{\beta A_1}{A_1\alpha + 1} & -\gamma - \mu - \theta & c_2 & 0 & 0 \\ 0 & \gamma & -\mu - \theta_1 - \pi & 0 & 0 \\ 0 & 0 & \pi & -\mu - \theta_2 - \omega & 0 \\ 0 & 0 & 0 & \omega & -\mu \end{pmatrix}$$

with

$$c_1 = -\frac{\beta\Lambda}{(\alpha\mu + \beta)A_1 + \mu} + \frac{\beta\Lambda A_1\alpha}{(A_1\alpha + 1)\left((\alpha\mu + \beta)A_1 + \mu\right)}$$

$$c_2 = \frac{\beta\Lambda}{(\alpha\mu + \beta)A_1 + \mu} - \frac{\beta\Lambda A_1\alpha}{(A_1\alpha + 1)\left((\alpha\mu + \beta)A_1 + \mu\right)}$$

$$A_1 = \frac{(\mu + \theta_1 + \pi)(\Re_0 - 1)\mu}{(\alpha\mu + \beta)(\mu + \theta_1 + \pi)}$$

The characteristic equation of this matrix is:

$$(\lambda + \mu + \theta_2 + \omega)(\lambda + \mu)f_1(\lambda) = 0$$

with $f_1(\lambda) = b_0\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3$

- $b_0 = (\alpha\mu + \beta)\alpha A^2 + 2(\alpha\mu + \beta/2)A + \mu$
- $b_1 = \left((\alpha\mu + \beta)^2 + (\alpha\mu + \beta)\alpha(\gamma + \mu) + (\alpha\mu + \beta)\alpha(\mu + \theta_1 + \pi)\right)A^2 + 2\left(\mu^2\alpha + \beta\mu + (\alpha\mu + \beta/2)(\gamma + \mu) + (\alpha\mu + \beta/2)(\mu + \theta_1 + \pi)\right)A + 3\mu^2 + (\theta_1 + \pi + \gamma)\mu$
- $b_2 = \left((\alpha\mu + \beta)^2(\gamma + \mu) + \left((\alpha\mu + \beta)^2 + (\alpha\mu + \beta)\alpha(\gamma + \mu)\right)(\mu + \theta_1 + \pi)\right)A^2 + 2\left((\alpha\mu^2 + \beta\mu)(\gamma + \mu) + \left(\mu^2\alpha + \beta\mu + (\alpha\mu + \beta/2)(\gamma + \mu)\right)(\mu + \theta_1 + \pi)\right)A + \mu^3 + (\theta_1 + \pi + \gamma)\mu^2 + \gamma(\theta_1 + \pi)\mu - \Re_0\mu(\gamma + \mu)(\mu + \theta_1 + \pi) + \mu^2(2\mu + \pi + \gamma + \theta_1)$
- $b_3 = \left(\left(A^2\mu^2\alpha^2 + 2A(A\beta\mu + \mu^2)\alpha + A^2\beta^2 + 2\beta\mu A - \mu^2(\Re_0 - 1)\right)\gamma + \left(A^2\mu^2\alpha^2 + 2A(A\beta\mu + \mu^2)\alpha + A^2\beta^2 + 2\beta\mu A - \mu^2(\Re_0 - 1)\right)\mu\right)\theta_1 + \left(\left(A^2\mu^2\alpha^2 + 2A(A\beta\mu + \mu^2)\alpha + A^2\beta^2 + 2\beta\mu A - \mu^2(\Re_0 - 1)\right)\pi + \left(A^2\mu^2\alpha^2 + 2A(A\beta\mu + \mu^2)\alpha + A^2\beta^2 + 2\beta\mu A - \mu^2(\Re_0 - 1)\right)\mu\right)\gamma + \mu\left(\left(A^2\mu^2\alpha^2 + 2A(A\beta\mu + \mu^2)\alpha + A^2\beta^2 + 2\beta\mu A - \mu^2(\Re_0 - 1)\right)\pi + A^2\alpha^2u^3 + 2\mu A(A\beta\mu + \mu^2)\alpha + A^2\beta^2\mu + 2A\beta\mu^2 - \mu^3(\Re_0 - 1)\right)$

The Hurwitz matrix for the polynomial $f_1(\lambda)$ is:

$$\begin{pmatrix} b_1 & b_0 & 0 \\ b_3 & b_2 & b_1 \\ 0 & 0 & b_3 \end{pmatrix}$$

We have:

$H_1 = |b_1| > 0$ and $H_2 = b_1b_2 - b_0b_3 > 0$, hence

$$H_3 = b_3H_2 > 0 \Leftrightarrow \Re_0 > 1$$

Therefore, the equilibrium point E_2 is locally asymptotically stable if and only if all roots of the characteristic polynomial $f_1(\lambda)$ have negative real parts, which holds if and only if $\Re_0 > 1$ ■

The stability of both the drug-free equilibrium point and the drug-addicted equilibrium point changes from stable to unstable, or vice versa, when $\Re_0 = 1$. The bifurcation parameter selected is $\beta^* = \beta$. By solving $\Re_0 = 1$, we have:

$$\Re = 1 = \frac{(\beta\,\Lambda\,\gamma)}{(\mu\,(\gamma+\mu\,)\,(\mu+\theta 1\,+\,\pi))}$$

$$\Leftrightarrow \beta^* = \frac{\mu(\gamma\,+\mu)(\mu\,+\,\theta_1\,+\pi)}{(\Lambda\gamma)}$$

It can be observed that if $\beta^* < \beta$, then the drug-free equilibrium E_1 is locally asymptotically stable, and E_2 is unstable. Conversely, if $\beta^* > \beta$, then E_1 is unstable, while E_2 is locally asymptotically stable.

To determine the type of bifurcation in System (1), the theorem presented by [19] is utilized. Consider the Jacobian matrix evaluated at (E_1, β^*):

$$J_{(E_1,\beta^*)} = \begin{pmatrix} -\mu & 0 & -\frac{\beta^*\Lambda}{\mu} & 0 & 0 \\ 0 & -k_1 & \frac{\beta^*\Lambda}{\mu} & 0 & 0 \\ 0 & \gamma & -\mu-\theta_1-\pi & 0 & 0 \\ 0 & 0 & \pi & -\mu-\theta_2-\omega & 0 \\ 0 & 0 & 0 & \omega & -\mu \end{pmatrix}$$

where the eigenvalues are $\lambda_1 = 0$ and the others are negative: $\lambda_{2,3} = -\mu$, $\lambda_4 = -(\gamma + 2\mu + \pi + \theta_1)$, $\lambda_5 = -(\mu + \theta_2 + \omega)$. We proceed to determine the left eigenvector (v) and right eigenvector (w) of $J_{(E_1,\beta^*)}$ corresponding to $\lambda = 0$. These are:

$$w = \left[0,0,(\mu\,+\theta_2\,+\omega)\frac{\mu}{\omega\pi}, c_3, 0\right]$$

and

v such that $v.w = 1$, is

$$v = \left[0, \frac{\omega\pi\gamma}{(\mu\,+\theta_2+\omega)\mu(\gamma\,+\mu)}, \frac{\omega\pi}{(\mu\,+\theta_2\,+\omega)mu}, 0, 0\right]$$

$$c_3 = \frac{(\gamma\,+\mu\,+\pi\,+\theta_1\,-\theta_2\,-\omega)(\gamma\,+\mu\,+\pi\,+\theta_1)}{\omega\pi}$$

Next, constants a and b are calculated as follows:

$$a = \sum_{h,i,j}^{5} v_h w_i w_j \frac{\partial f_h}{\partial y_i \partial y_j}(E_0, \beta^*)$$

$$= v_2 w_3 w_3 \frac{\partial f_2}{\partial y_3 \partial y_3}\,(E_0, \beta^*)$$

$$= c_4\left(-\frac{2(\gamma\,+\mu)(\mu\,+\,\theta_1\,+\pi)\alpha}{\gamma}\right)$$

$$= -\frac{2(\mu\,+\theta_2\,+\omega)\mu(\mu\,+\theta_1\,+\pi)\alpha}{\omega\pi} < 0$$

with

$$c_4 = \frac{\omega\pi\gamma}{(\mu\,+\theta_2+\omega)\mu(\gamma\,+\mu)}\frac{(\mu\,+\theta_2\,+\omega)\mu}{\omega\pi}\frac{(\mu\,+\theta_2\,+\omega)\mu}{\omega\pi}$$

and

$$b = \sum_{h,i}^{5} v_h w_i \frac{\partial f_h}{\partial y_i \partial \beta}(E_0, \beta^*)$$

$$= v_2 w_3 \frac{\partial f_2}{\partial y_3 \partial \beta}(E_0, \beta^*)$$

$$= \left(\frac{\gamma}{\gamma+\mu}\right)\frac{\Lambda}{\mu} = \frac{\gamma\Lambda}{\mu(\gamma+\mu)} > 0$$

where the variables correspond to $y_1 = S$, $y_2 = E$, $y_3 = A$, $y_4 = Q$, and $y_5 = R$. Thus, the conditions $a < 0$ and $b > 0$ hold when $\beta = \beta^*$. According to Theorem 4.1 in [19], System (1) undergoes a forward bifurcation at $\Re = 1$

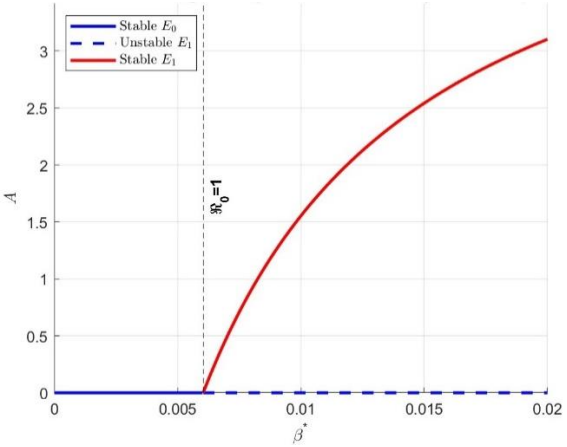

Figure 2. Forward Bifurcation when $\Re_0 = 1$

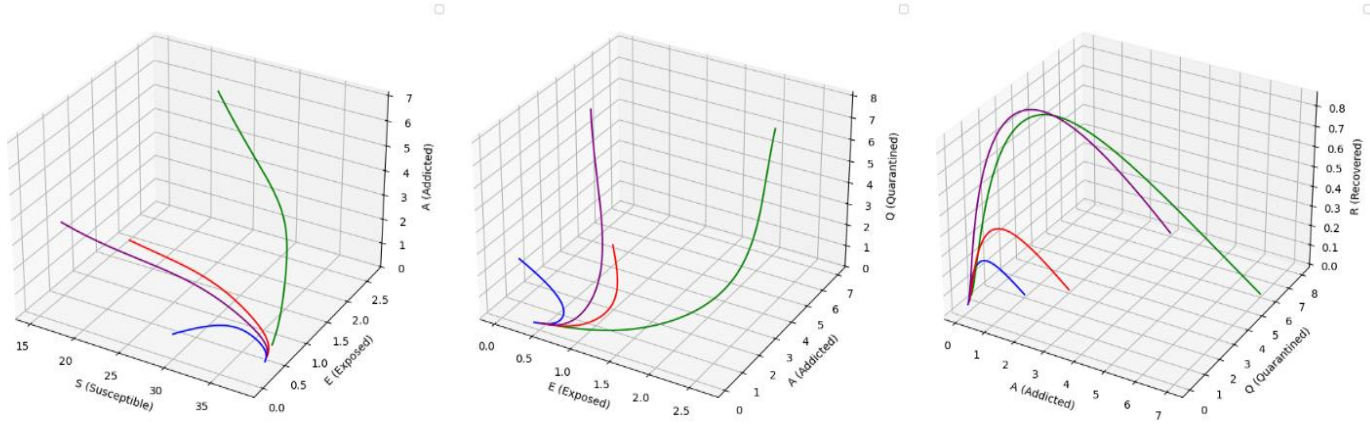

Figure 3. Phase Portrait of Drug-Free Dynamics

Figure 2 illustrates the change in stability within the dynamics of the addiction model. At the point $\beta^* \approx 0.005$, a significant shift occurs in the system. To the left of β^*, the curve for A remains close to zero, indicating that the number of addicted individuals stays low or there is no addiction within the population. After crossing the point β^*, the number of addicts A begins to increase sharply, showing that the spread of drug addiction becomes easier and produces more addicted individual. The forward bifurcation depicted in this graph represents a profound change in the dynamic system of drug addiction: when β reaches the critical value β^*, the system transitions from a stable state of very low or no addiction to a state where addiction increases rapidly and stabilizes at a higher level. This demonstrates that an increasing transmission rate (spread of illegal drugs) can cause addiction numbers in the population to surge, becoming a more significant problem.

F. Numerical Simulation

This section discusses the dynamics of System (1) using the parameter values presented in Table 1. For $\Lambda = 1.2$, the basic reproduction number is found to be 0.9936, according to the theorem 1 on the drug-free equilibrium point E_1 which is locally asymptotically stable. See Figure 3, at the start of the simulation, the number of exposed individuals $E(t)$ decreases after some time; although exposed, these individuals may not yet be fully addicted but still have the potential to progress into addiction without intervention. As the number of exposed individuals declines, the number of addicted individuals $A(t)$ will eventually decrease over time as well. Correspondingly, the numbers of individuals undergoing rehabilitation $Q(t)$ and those recovered $R(t)$ also decrease because the population of addicted individuals diminishes. The trajectories from different initial conditions converge toward a state with minimal or no addiction. The model predicts that under these conditions, the population will stabilize with a high number of susceptible and recovered individuals and very low exposed, addicted, or quarantined individuals.

Next, for $\Lambda = 30$, we obtain $\Re_0 = 24.84 > 1$. According to the Theorem 2, the drug addiction equilibrium point is locally asymptotically stable. Figure 4 with $\alpha = 0.5$ shows that initially, $S(t)$ increases, but over time, its number decreases as individuals move into the $E(t)$ class and approach a stable level. The graph of $E(t)$ rises sharply after some time as susceptible individuals become influenced by drug use; after reaching a certain point, this growth slows and then stabilizes. With a high number of exposed individuals, it is very likely that many also begin to experience drug addiction. Due to low awareness and education about the dangers of addiction, the transition rate from $E(t)$ to $A(t)$ is also high. The graph of $Q(t)$ increases rapidly, reflecting many people entering treatment or rehabilitation. With proper rehabilitation processes, the chances of recovery and freedom from drug addiction increase. The system dynamics reflect the progression of drug addiction through stages: susceptibility, exposure, addiction, rehabilitation, and recovery. Since, addiction spreads and grows initially in the population. Rehabilitation efforts (quarantine and recovery compartments) play a crucial role in reducing addicted individuals over time.

A graph is presented with several variations of the parameter α. The figure illustrates the impact of α on different population groups within the dynamics of drug addiction. In the Figure 5, as the value of α increases, the number of exposed individuals decreases more rapidly. This indicates the effectiveness of interventions that prevent susceptible individuals from being influenced to use illegal drugs. In the Figure 6, the parameter α indirectly suppresses the number of individuals becoming addicted to drugs. Increased intervention effectiveness reduces the likelihood of people being influenced by drug use. The parameter α can be considered a factor that modulates the transmission rate based on the number of addicted individuals. The larger the α, the greater its influence in reducing the transmission rate when the addicted population increases. In this context, α regulates the extent to which addicted individuals can transmit addiction to susceptible individuals. When α increases, the transmission is hindered even as the number of addicted individuals grows.

The parameter α can represent the presence of interventions that reduce the effectiveness of addiction transmission. For instance, public health interventions such as prevention programs or education about the dangers of drug abuse can increase α, which ultimately reduces the transmission rate.

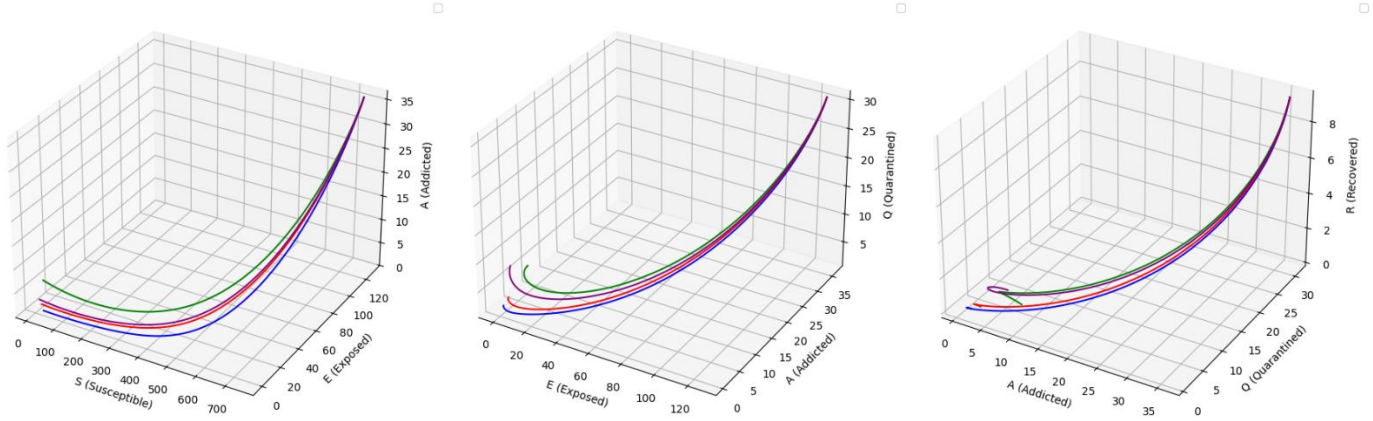

Figure 4. Phase Portrait of Drug Addiction Dynamics

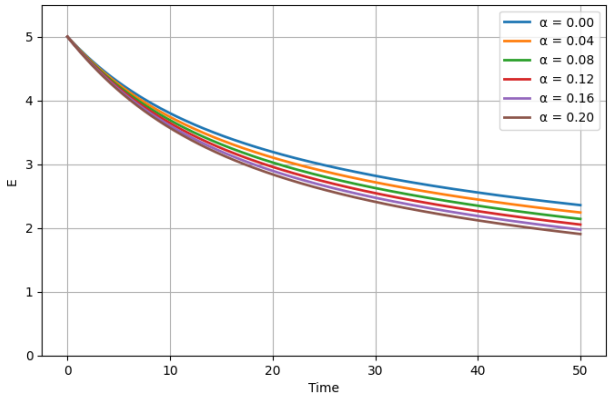

Figure 5. Graph of the Effect of Various α on Compartment *E*

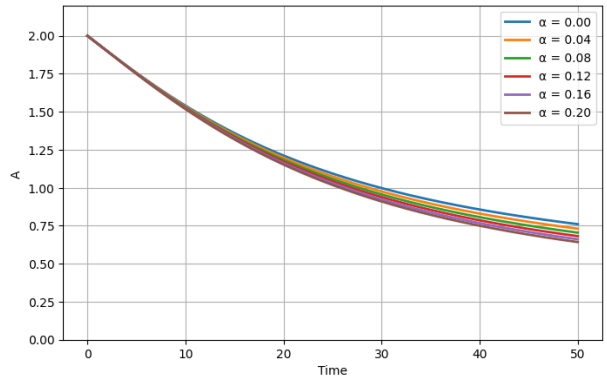

Figure 6. Graph of the Effect of Various α on Compartment *A*

III. CONCLUSSION AND SUGGESTION

This study discusses the dynamics of drug addiction with a nonlinear incidence rate. There are two equilibrium points based on the basic reproduction number $\Re_0$. When $\Re_0 \leq 1$, a drug-free equilibrium point is obtained. This point is locally asymptotically stable when R0<1, whereas when $\Re_0 > 1$, there is a drug addiction equilibrium point that is asymptotically stable. A forward bifurcation phenomenon also occurs when $\Re_0 = 1$, which results in a change in the stability condition of the equilibrium point. Numerical simulations demonstrate the effect of the parameter α. This parameter can be considered as an intervention to limit the number of individuals addicted to drugs, even though many individuals have already become addicted. Such interventions may include education about the dangers of drug abuse. The parameter α can be regarded as an indicator of an individual's "resistance" to addiction after exposure, which becomes increasingly significant as the number of addicted individuals grows. In certain situations, if α is sufficiently large, the transmission can slow down significantly despite the presence of many addicted individuals.

Many aspects have not been discussed in this study to simplify the analysis. The current discussion does not include relapse conditions for recovered addicts. This relapse condition remains a possibility for individuals who have recovered from drug addiction. Future research could consider including relapse conditions. Individuals who are recovering and have been declared cured still face the risk of returning to drug use.

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