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Hasse diagrams of basic blocks containing four comparable reducible elements and having nullity six

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Abstract

In 2020, Bhavale and Waphare introduced the concepts of fundamental basic block and basic block. Further, Bhavale and Waphare have provided the recursive formulae of the number of non-isomorphic fundamental basic blocks as well as basic blocks, containing r comparable reducible elements and having nullity l. With the help of those formulae, Bhavale et al. obtained the Hasse diagrams of all non-isomorphic basic blocks containing up to four comparable reducible elements and having nullity up to five. In this paper, we actually obtain the Hasse diagrams of the basic blocks containing four comparable reducible elements and having nullity six.

Keywords: Poset, Lattice, Chain, Counting.
MSC Classification 2020: 06A05, 06A06, 06A07.

1 Introduction

Birkhoff [7] in 1940 raised the open problem of enumerating all posets/lattices which are uniquely determined up to isomorphism by their diagrams, considered purely as graphs. Many authors all over the world attempted this problem. In 1979, Kyuno [12] introduced an inductive algorithm for constructing finite lattices, through which he was able to generate Hasse diagrams of lattices containing up to nine elements, classified up to isomorphism. Later, in 1989, Kolhe [11] proposed a more refined algorithm to determine the total number of non-isomorphic lattices with 8 and 9 elements. According to his result, there are 1082 distinct non-isomorphic lattices on 9 elements, a count that differs slightly from the figure 1078 reported in [10] for the same case. Recently, Monteiro et al. [13] obtained Hasse diagrams of posets with up to 7 elements, and the number of posets with 10 elements, without the use of computer programs. For more details on enumerations of posets and lattices, reader may refer [8] and [10].

In 2020, Bhavale and Waphare [6] introduced the concepts of fundamental basic block and basic block. Further, they have provided the recursive formulae of the number of non-isomorphic fundamental basic blocks as well as basic blocks, containing r comparable reducible elements and having nullity l. With the help of those formulae Bhavale et al. obtained the Hasse diagrams of all basic blocks containing up to four comparable reducible elements and having nullity up to nullity five (see [1],[2],[3],[5], and [4]).

2 Preliminaries

An element a in a lattice L is meet-reducible (join-reducible) in L if there exist $b, c \in L$ both distinct from a, such that $inf\{b,c\} = a(sup\{b,c\} = a)$. An element a in a lattice L is said to be reducible if it is either meet-reducible or join-reducible. a is said to be meet-irreducible (join-irreducible) if it is not meet-reducible(join-reducible). a is doubly irreducible if it is both meet-irreducible and join-irreducible. A finite lattice of order n is called dismantlable if there exists a chain $L_1 \subset L_2 \subset \cdots L_n(= L)$ of sublattices of L such that $|L_i| = i$, for all i (see [14]).

An element x of a poset P is said to be doubly irreducible if it has at most one upper cover and at most one lower cover in P. Let $Irr(P)$ denotes the set of all doubly irreducible elements of P. The nullity of a poset P is defined as the nullity of its cover graph. A poset P is a basic block if it is one element or $Irr(P) = \emptyset$ or removal of any doubly irreducible element reduces nullity by one (see [6]). A dismantlable lattice F is said to be a fundamental basic block if it is a basic block and all the adjunct pairs in the adjunct representation of F into chains are distinct (see [6]). For the other definitions, notation and terminology; reader may refer [9] and [15].

3 Counting of basic blocks

Bhavale and Waphare [6] obtained the formulae of counting of all non-isomorphic fundamental basic blocks and basic blocks containing r comparable reducible elements and having nullity k. Let $\mathscr{F}_r(l)$ be the class of all non-isomorphic fundamental basic blocks containing r comparable reducible elements and having nullity l. Note that $\mathscr{F}_0(0)$ consists of a 1-chain only. $|\mathscr{F}_2(1)| = 1$. The following results are due to Bhavale and Waphare [6].

Theorem 3.1. *[6] For fixed $r \geq 1$, and for $\lfloor \frac{r+2}{2} \rfloor \leq l \leq \binom{r+1}{2}$,*

$$|\mathscr{F}_{r+1}(l)| = \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} |\mathscr{F}_{r-j}(l-k)| .$$

Using Theorem 3.1, we have $|\mathscr{F}_4(2)| = 3,\ |\mathscr{F}_4(3)| = 16,\ |\mathscr{F}_4(4)| = 15,$ $|\mathscr{F}_4(5)| = 6$ and $|\mathscr{F}_4(6)| = 1$.

Let $\mathscr{B}_r(l)$ be the class of all non-isomorphic basic blocks containing r comparable reducible elements and having nullity l.

Theorem 3.2. *[6] For fixed $r \geq 2$, and for $\lfloor \frac{r+2}{2} \rfloor \leq m \leq l \leq \binom{r+1}{2}$,*

$$|\mathscr{B}_r(l)| = \sum_{m=\lfloor \frac{r+1}{2} \rfloor}^{l} \binom{l-1}{m-1} |\mathscr{F}_r(m)|.$$

Using Theorem 3.2, we have $|\mathscr{B}_4(6)| = (5)(3) + (10)(16) + (10)(15) + (5)(6) + (1)(1) = 15 + 160 + 150 + 30 + 1 = 356$.

4 Hasse diagrams of basic blocks

In this section, we provide up to isomorphism, Hasse diagrams of all 356 basic blocks containing four comparable reducible elements and having nullity six.

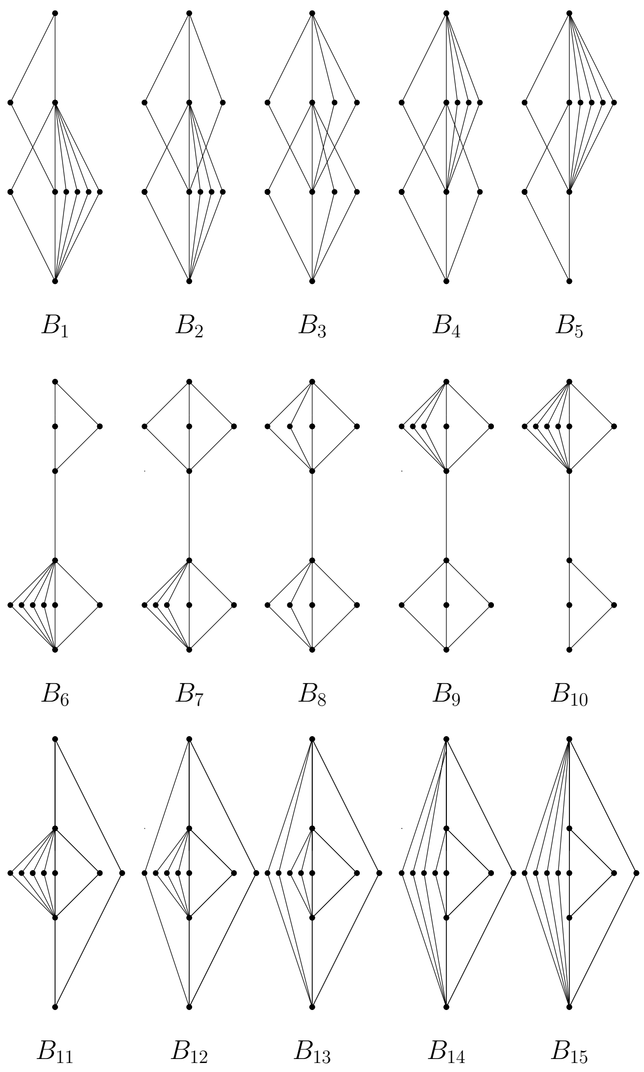
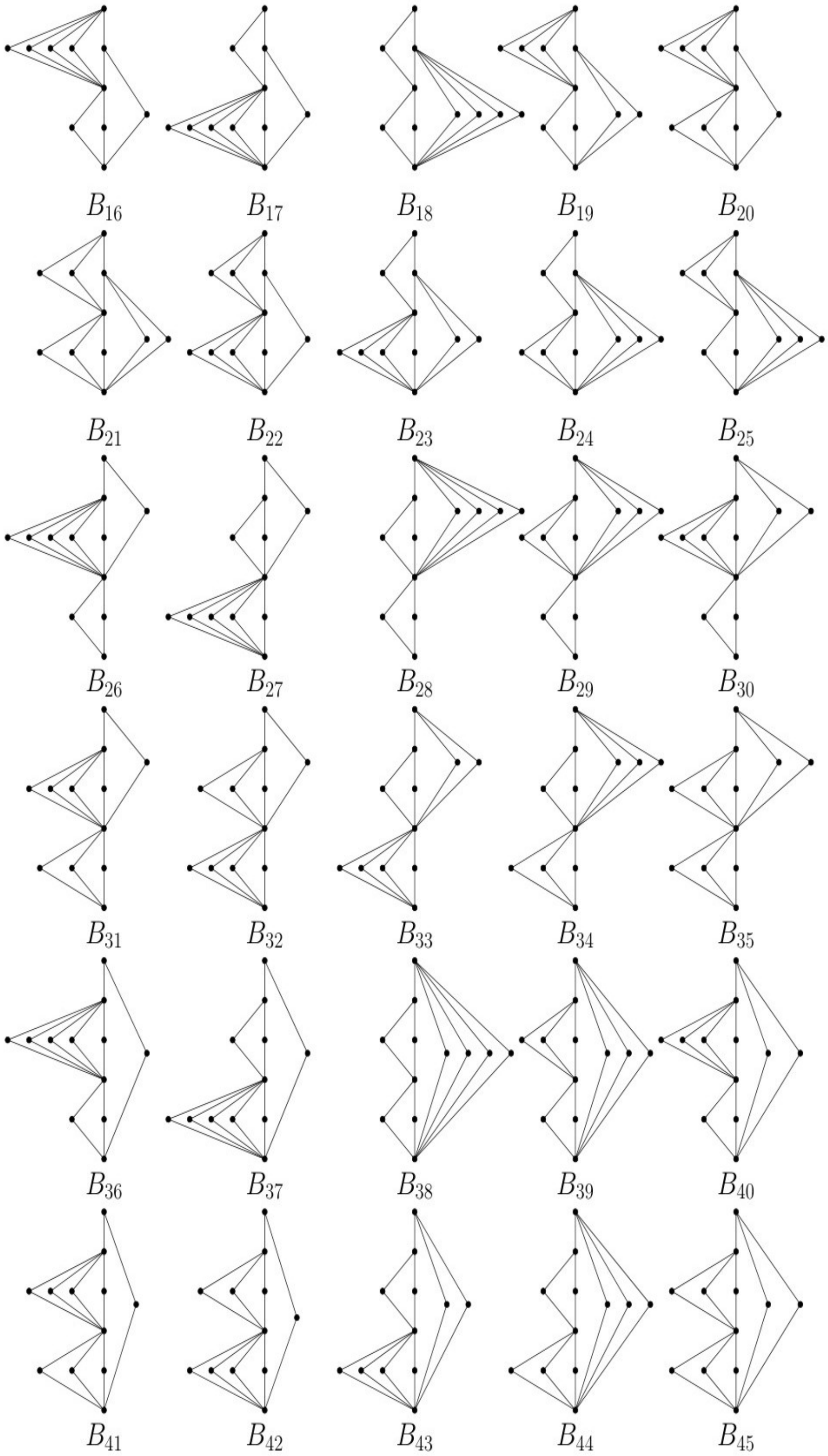
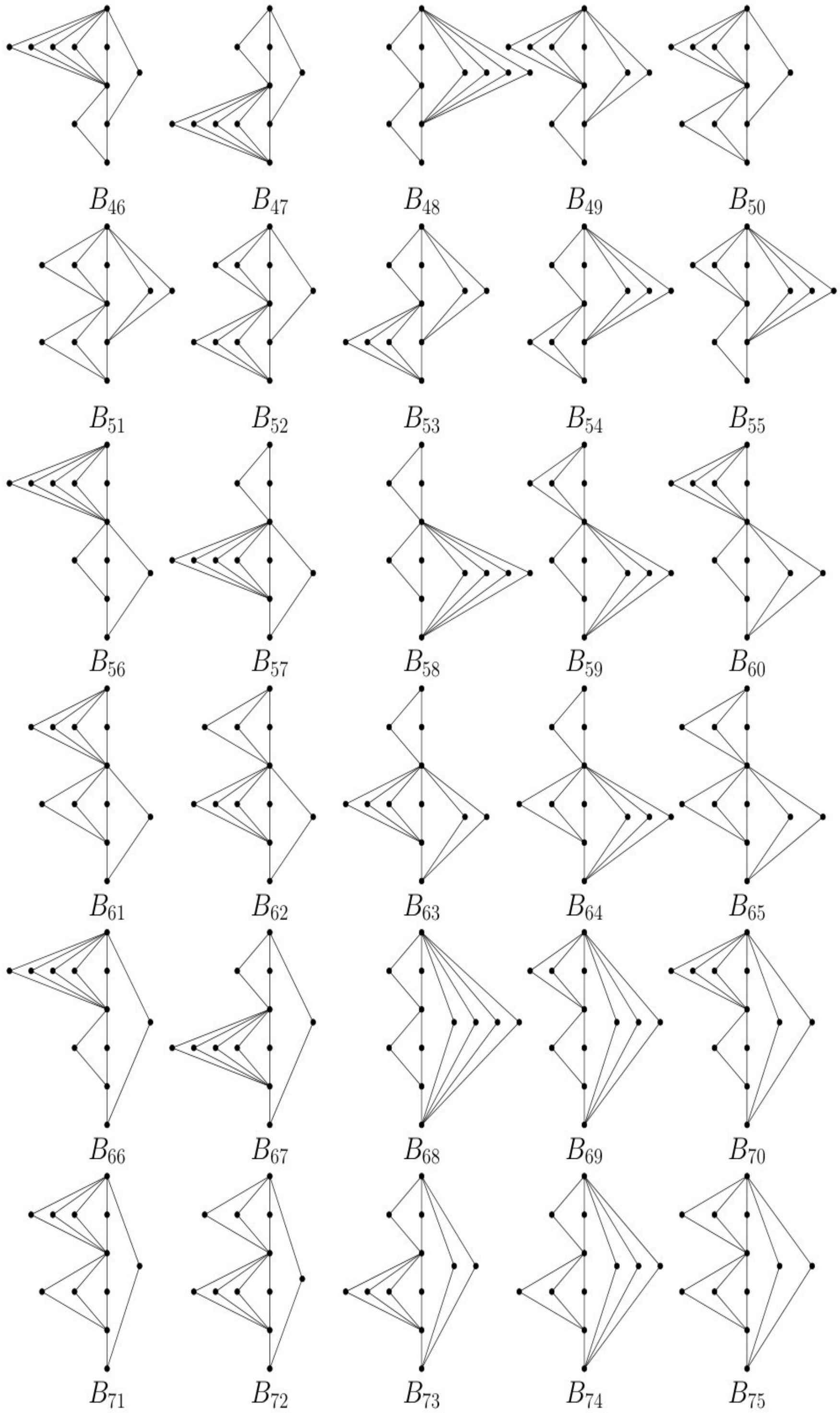
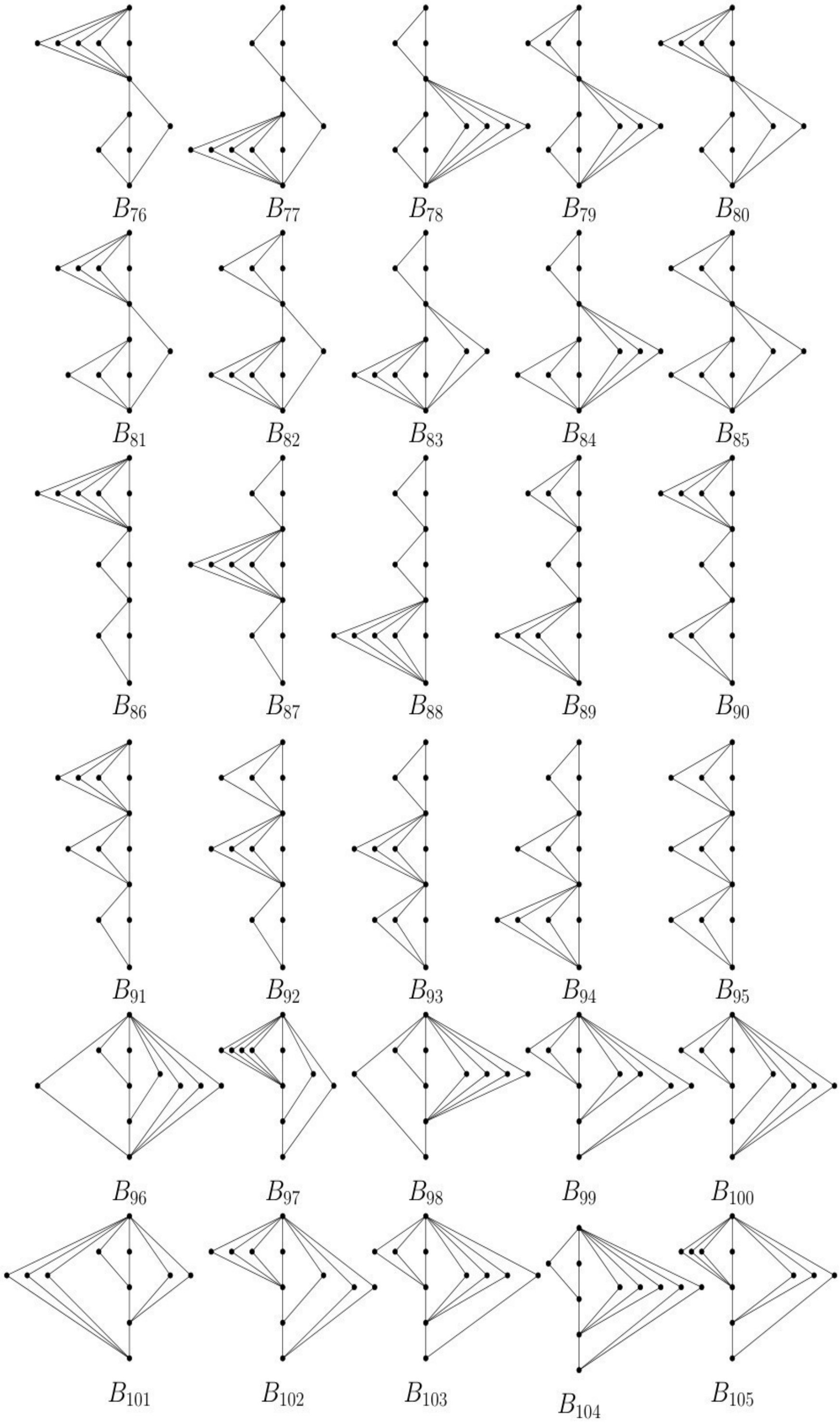


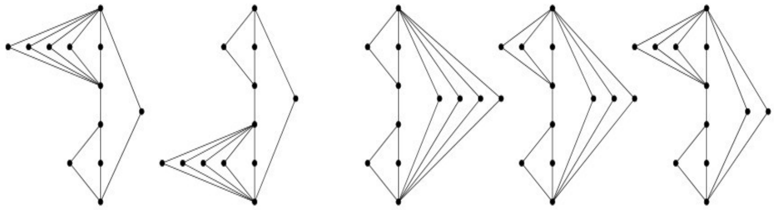
Figure I: Basic blocks obtained using 3 fundamental basic blocks of nullity 2.


B_{16} B_{17} B_{18} B_{19} B_{20}
B_{21} B_{22} B_{23} B_{24} B_{25}
B_{26} B_{27} B_{28} B_{29} B_{30}
B_{31} B_{32} B_{33} B_{34} B_{35}
B_{36} B_{37} B_{38} B_{39} B_{40}
B_{41} B_{42} B_{43} B_{44} B_{45}

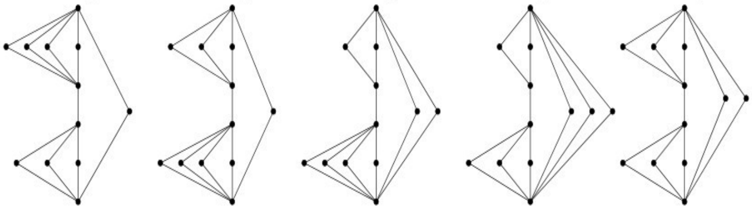

B_{46} B_{47} B_{48} B_{49} B_{50}
B_{51} B_{52} B_{53} B_{54} B_{55}
B_{56} B_{57} B_{58} B_{59} B_{60}
B_{61} B_{62} B_{63} B_{64} B_{65}
B_{66} B_{67} B_{68} B_{69} B_{70}
B_{71} B_{72} B_{73} B_{74} B_{75}

"Hasse diagrams of basic blocks containing four comparable reducible elements and having nullity"

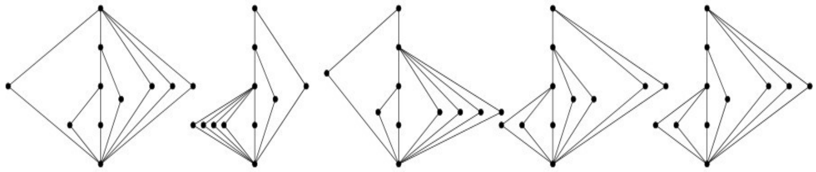




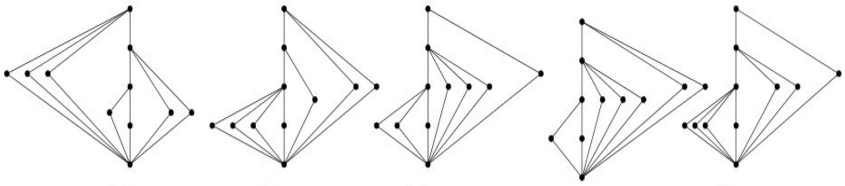
B_{106} B_{107} B_{108} B_{109} B_{110}



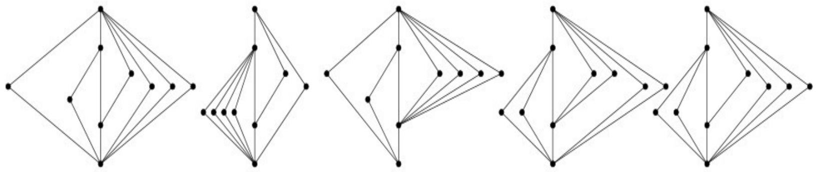
B_{111} B_{112} B_{113} B_{114} B_{115}



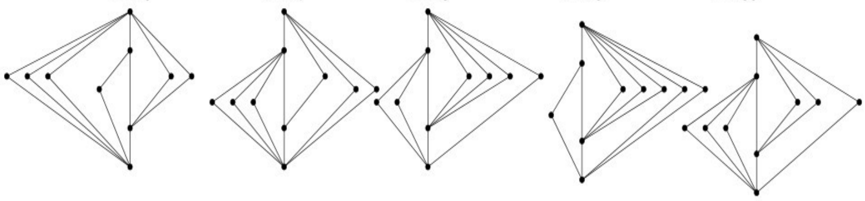
B_{116} B_{117} B_{118} B_{119} B_{120}



B_{121} B_{122} B_{123} B_{124} B_{125}



B_{126} B_{127} B_{128} B_{129} B_{130}



B_{131} B_{132} B_{133} B_{134} B_{135}

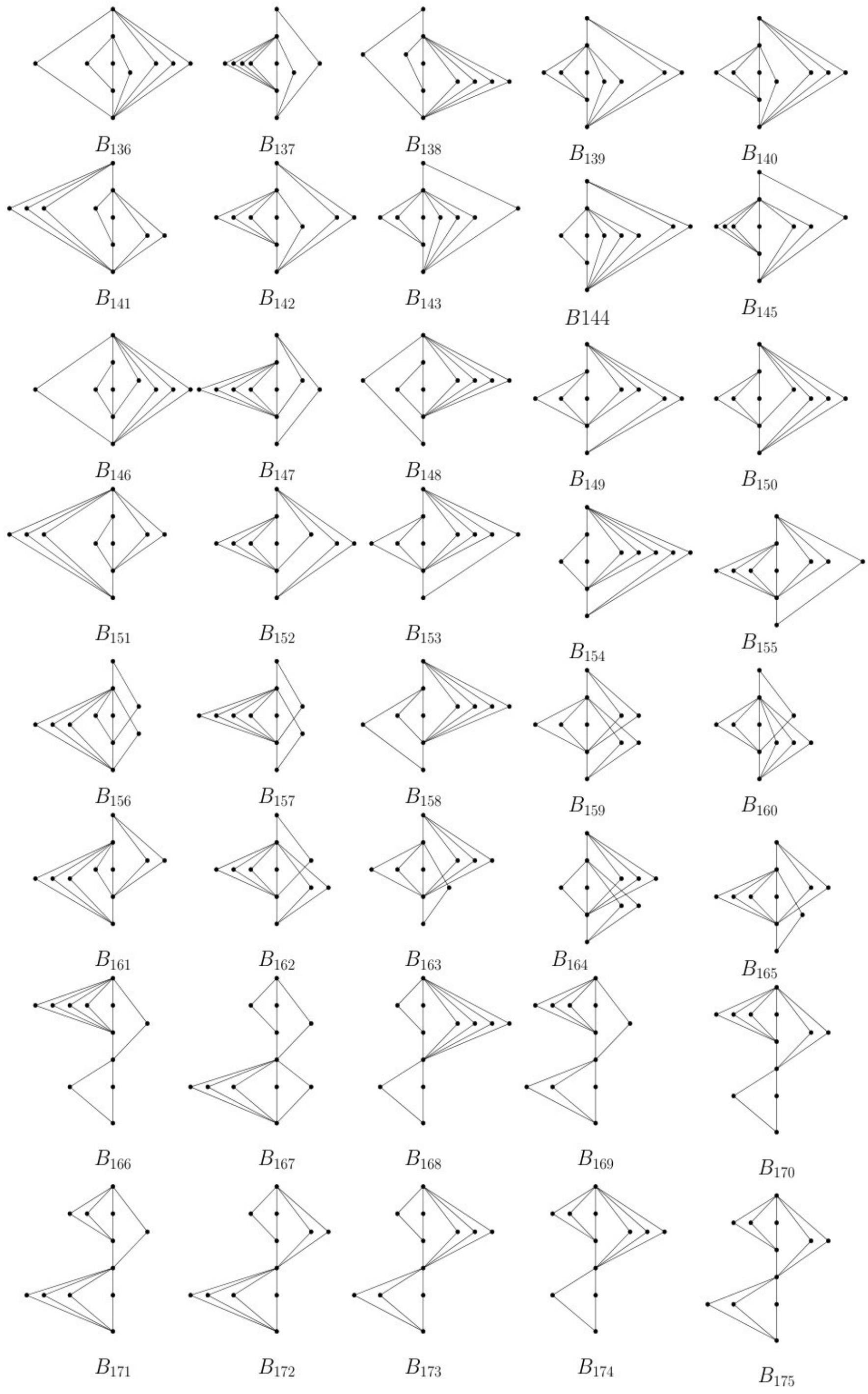
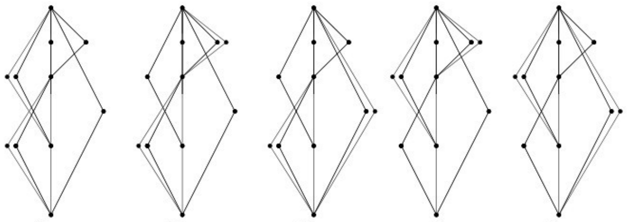
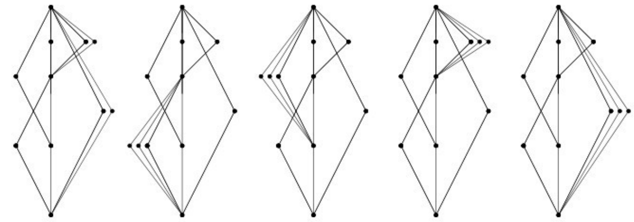


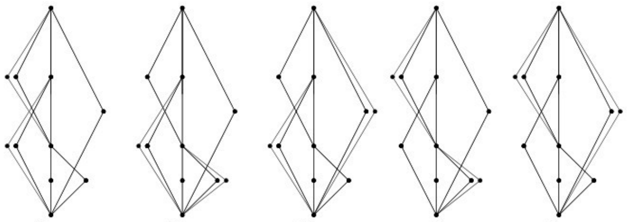
Figure II: Basic Blocks obtained using 16 fundamental basic blocks of nullity 3.



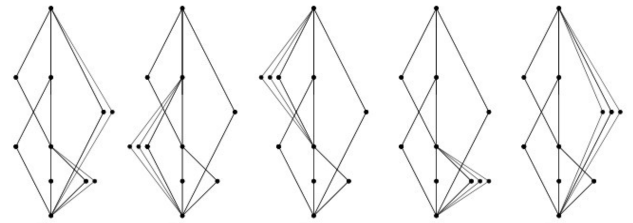
B_{176} B_{177} B_{178} B_{179} B_{180}



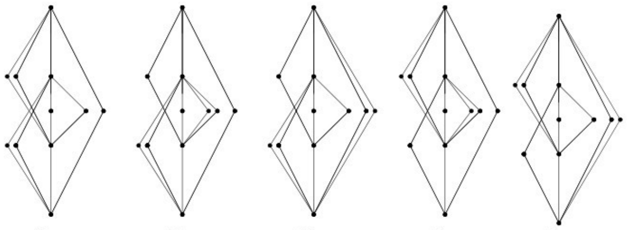
B_{181} B_{182} B_{183} B_{184} B_{185}



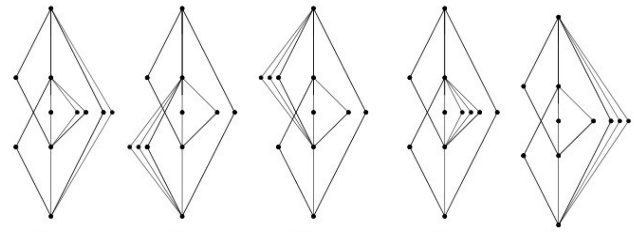
B_{186} B_{187} B_{188} B_{189} B_{190}



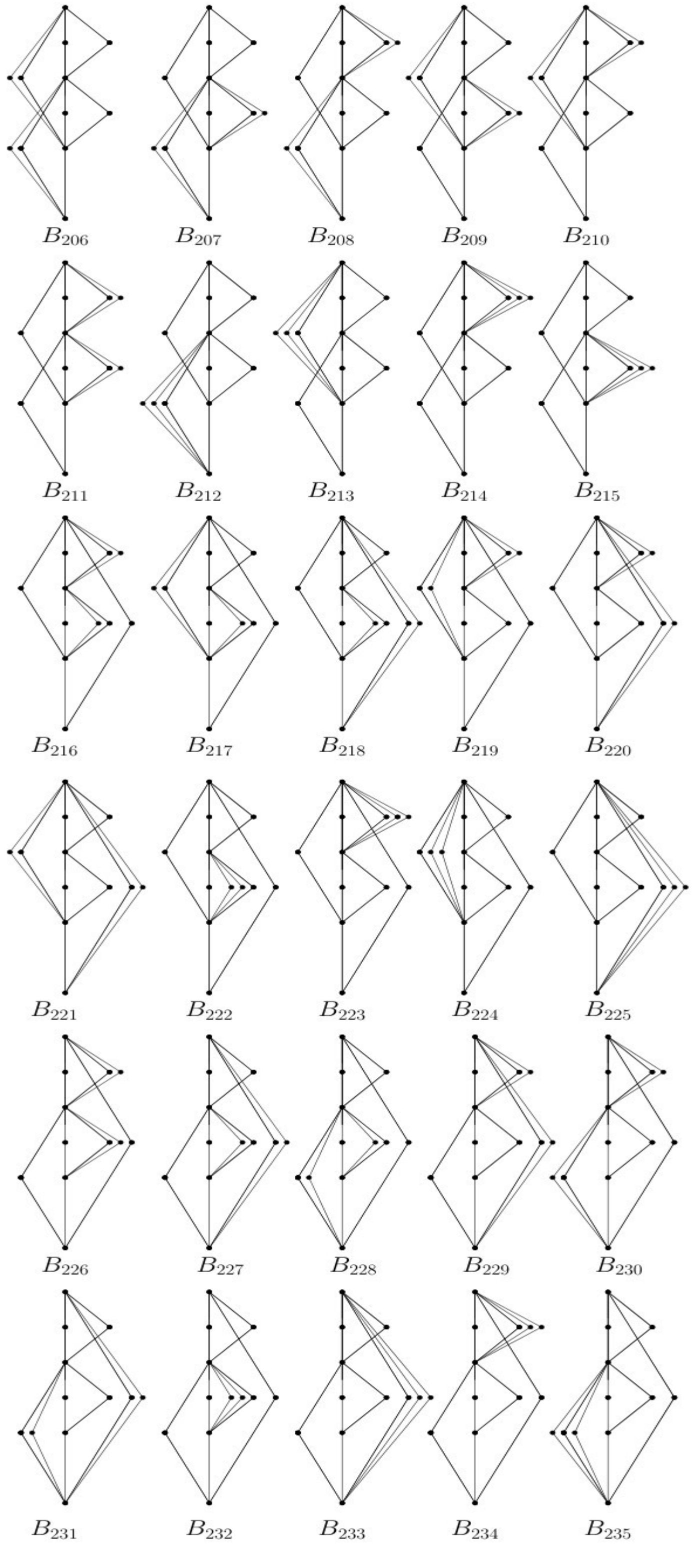
B_{191} B_{192} B_{193} B_{194} B_{195}



B_{196} B_{197} B_{198} B_{199} B_{200}



B_{201} B_{202} B_{203} B_{204} B_{205}


B_{206} B_{207} B_{208} B_{209} B_{210}
B_{211} B_{212} B_{213} B_{214} B_{215}
B_{216} B_{217} B_{218} B_{219} B_{220}
B_{221} B_{222} B_{223} B_{224} B_{225}
B_{226} B_{227} B_{228} B_{229} B_{230}
B_{231} B_{232} B_{233} B_{234} B_{235}

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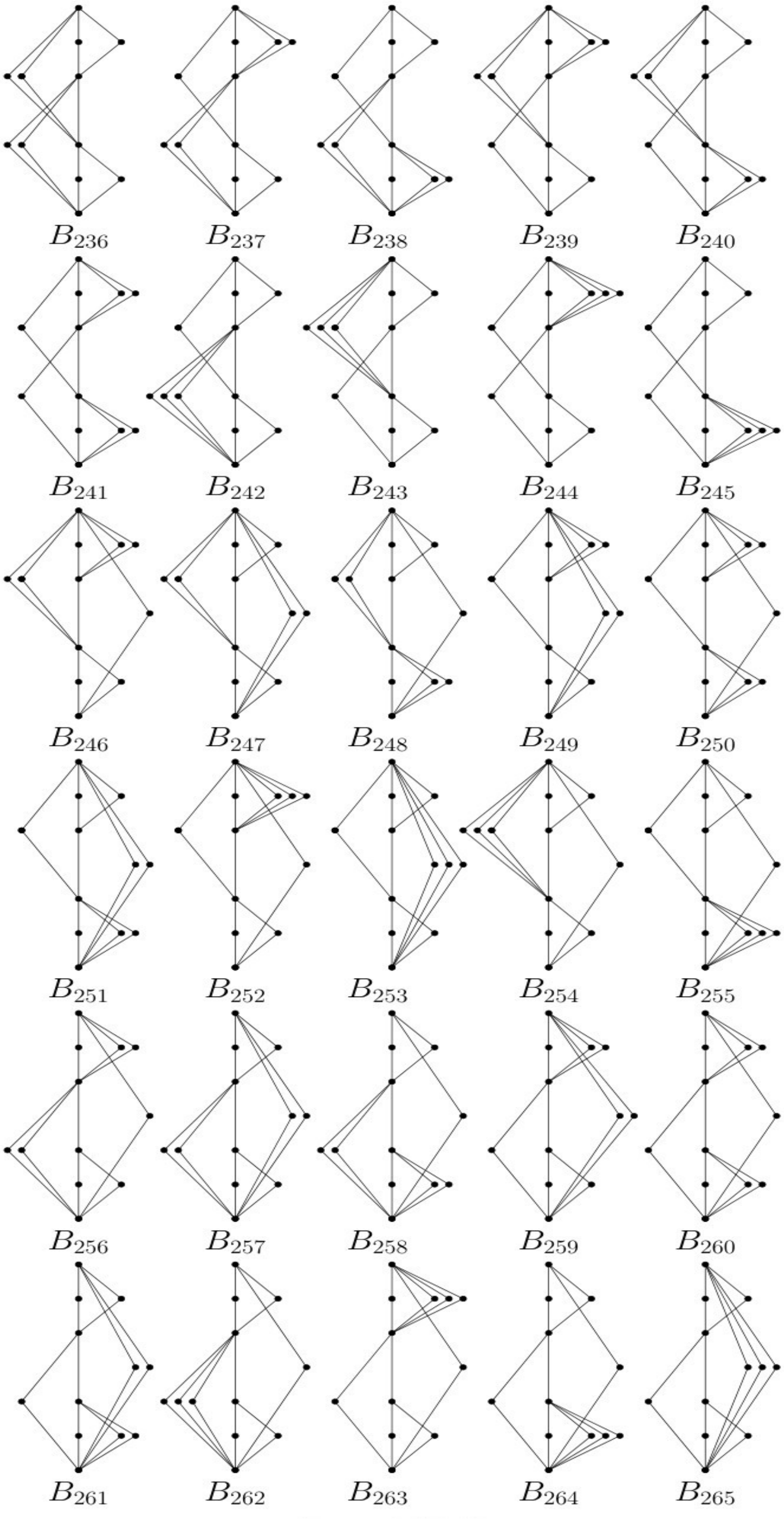
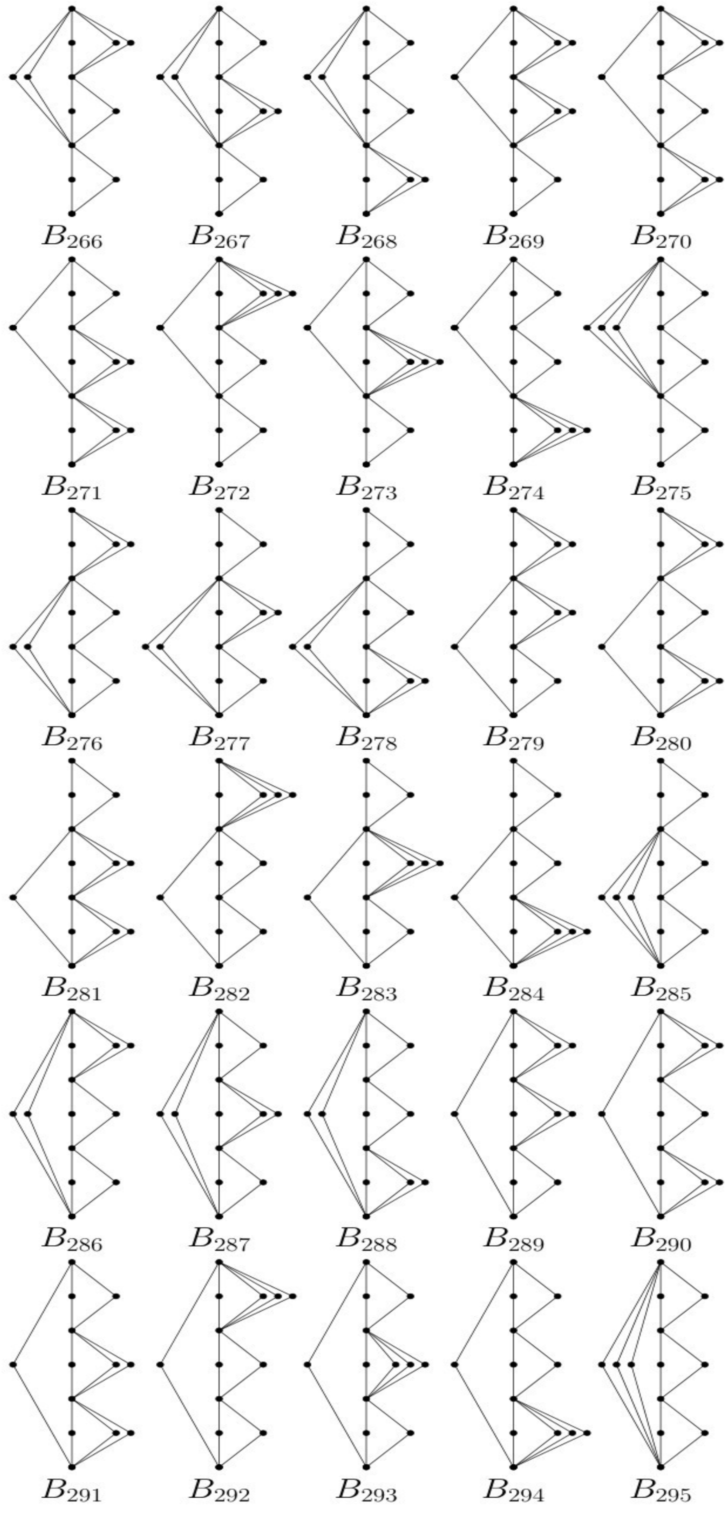



B_{266} B_{267} B_{268} B_{269} B_{270}
B_{271} B_{272} B_{273} B_{274} B_{275}
B_{276} B_{277} B_{278} B_{279} B_{280}
B_{281} B_{282} B_{283} B_{284} B_{285}
B_{286} B_{287} B_{288} B_{289} B_{290}
B_{291} B_{292} B_{293} B_{294} B_{295}

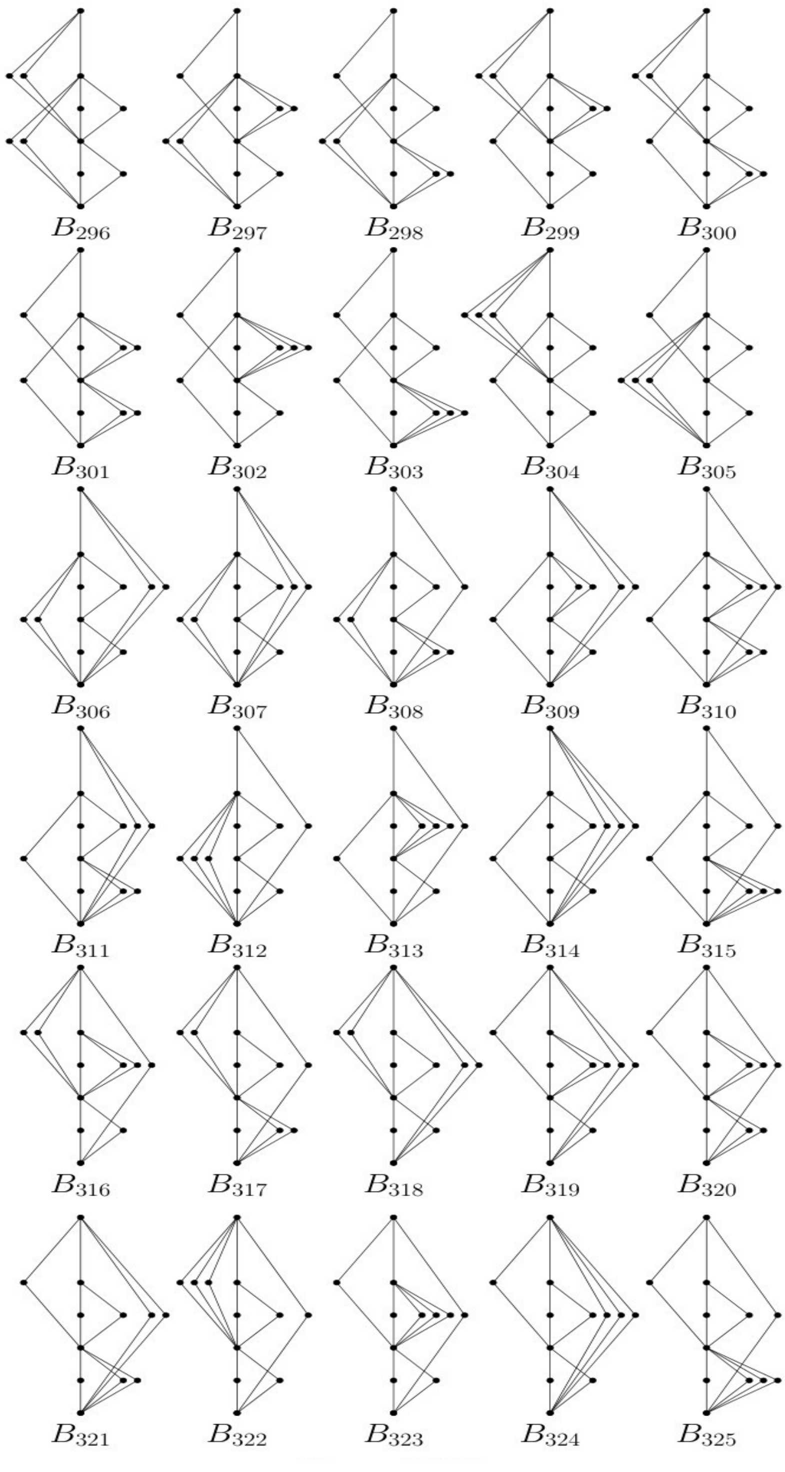


Figure III:Basic Blocks obtained using 15 fundamental basic blocks of nullity 4.

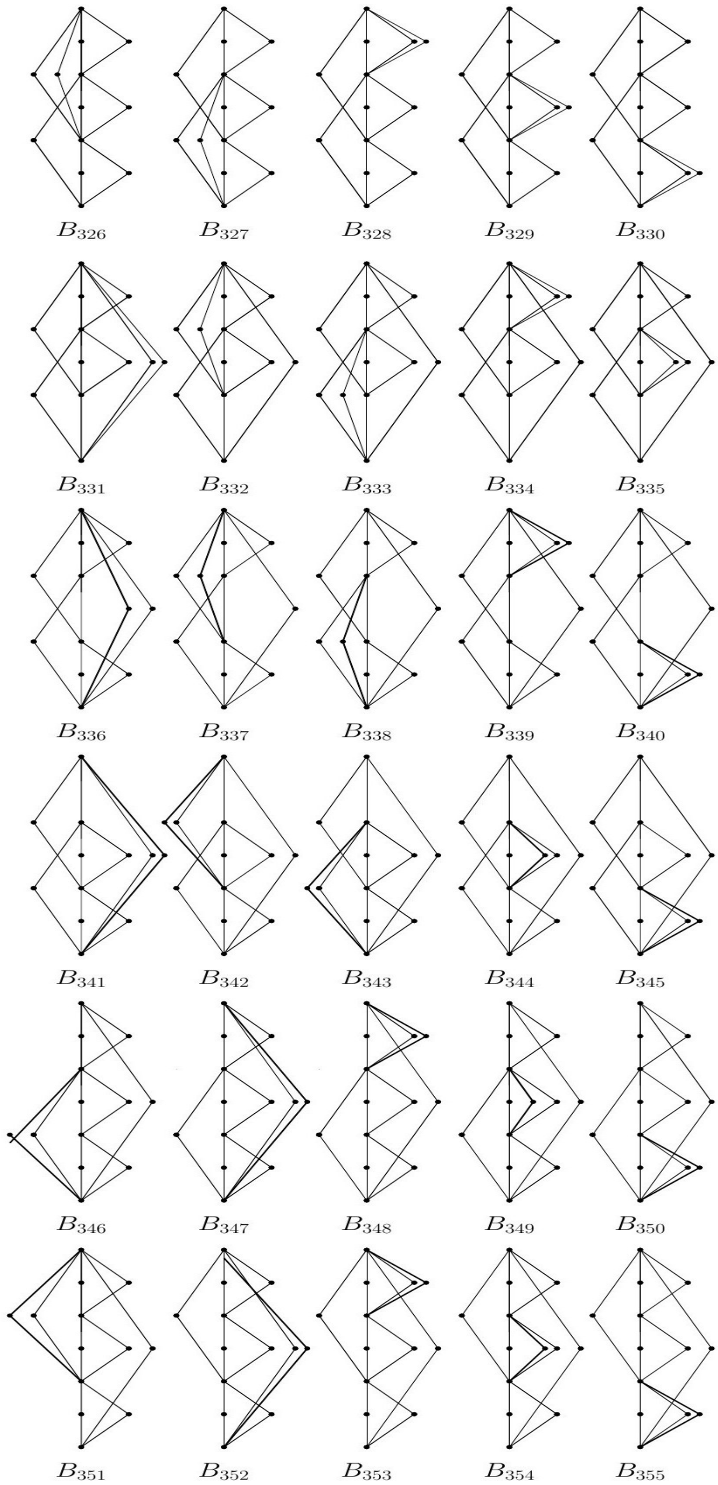
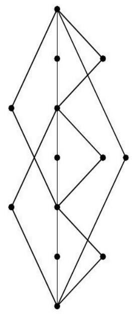


Figure IV:Basic Blocks obtained using 6 fundamental basic blocks of nullity 5.



B_{356}

Figure V: Basic Block (which is also the fundamental basic block) of nullity 6.

As far as the basic blocks containing 4 comparable reducible elements concerned, there are 3 basic blocks with nullity two, 22 basic blocks with nullity three, 72 basic blocks with nullity four, and 174 basic blocks with nullity five. Thus there are altogether 271 non-isomorphic basic blocks containing four comparable reducible elements with nullity up to five. In this paper, we obtain 356 non-isomorphic basic blocks containing four reducible elements having nullity six. So there are total 627 non-isomorphic basic blocks containing four comparable reducible elements, and having nullity up to six.

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