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Clean Modules Constructed from the External Direct Sum of Clean Modules

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ARTICLE INFO	ABSTRACT
Published Online: 30 July 2025 **Corresponding Author:** Dheva Ufiz 'Aliyah	Let $(R\ ,+,\bullet)$ be a ring with identity and M be an R-module. A ring in which every element can be expressed as the sum of an idempotent and a unit element is called a clean ring. An R-module M that is mapped to itself is called an endomorphism, denoted by $(End_R(M))$. The set of all endomorphisms forms a ring under addition and function composition. This fact motivates the notion of clean modules over a ring. An R-module M is called a clean module if $End_R(M)$ is a clean ring. This concept was first investigated by Camillo et al. In this article, we identify a special case of a clean module constructed from the external direct sum of two modules over the same ring. We show that if M_1 and M_2 are clean modules over R, then their external direct sum $M_1 \times M_2$ is also a clean module.

KEYWORDS: Clean Rings, Clean Modules, Strongly Clean Modules, Endomorphism.

I. INTRODUCTION

In an R-module, there is a special relationship between two different modules, which is expressed by a linear mapping called a homomorphism [1]. If a homomorphism maps two identical modules, it is called an endomorphism, denoted by $End_R(M)$. Furthermore, all sets of endomorphisms can form a ring structure with addition and function composition, denoted by $(End_R(M),+,\circ)$ [2][3].

In a ring, there are elements called unit elements $(U(R))$ and idempotent elements $(Id(R))$ [4]. Furthermore, there is a relationship between idempotent elements and unit elements in a ring. In 1976, Neumann introduced the definition of a regular Von Neumann ring, namely a ring whose every element can be described as the product of its unit and idempotent elements [5]. This motivated a new structure called a clean ring. In 1977, Nicholson introduced the concept of a clean ring, namely a ring whose every element can be described as the sum of its unit and idempotent elements [6]. The study of clean rings has been extended to clean semirings discussed by [7][8].

In an R-module M, if we replace M with R, then obtain a module over itself [9]. It is found that the ring R is isomorphic to the ring $End_R(R)$, as a result, if the ring R is a clean ring, then the ring $End_R(R)$ is also clean [10]. Then, it is generalized to the clean ring $End_R(M)$ [11][12]. The clean ring $End_R(M)$ underlies the emergence of the concept of a clean module, namely a module whose endomorphism ring is a clean ring [13]. Further studies on clean modules can be found in the research [14][15]. On the other hand, research on clean modules has been extended to clean comodules and clean coalgebras [16][17].

In a ring, the concept of a direct product is known. A direct product of the ring R is a clean ring if and only if the ring R is a clean ring [18][19]. In module theory, analog to this concept, if M_1 and M_2 are modules over the same ring, then we can construct a module $M_1 \times M_2$, which is called the external direct sum [20]. Based on this fact, a hypothesis is proposed that the external direct sum of modules is a clean module if and only if the modules are clean modules. In this study, it is proven that the external direct sum of modules forms a clean module. It is also proven that it holds for strongly clean modules.

II. METHOD

The method used in this study is a literature review, which involves examining various articles and references related to the concept of clean modules. This research investigates the conditions under which a module formed as the external direct sum of two modules over the same ring is a clean module. The procedure involves reviewing concepts from ring theory and module theory, particularly those related to Cartesian products in a ring, unit elements, idempotent elements, ring isomorphisms, as well as external direct sums in modules and module endomorphisms. After reviewing the literature, the study continues by investigating the cleanness property of external direct sums of modules. At this stage, the cleanness of the external direct sum formed from two modules is examined, and then generalized to the case of n modules.

This study focuses on left modules as the main object of investigation, denote by R-module M. The following figure presents the state of the art of this research.

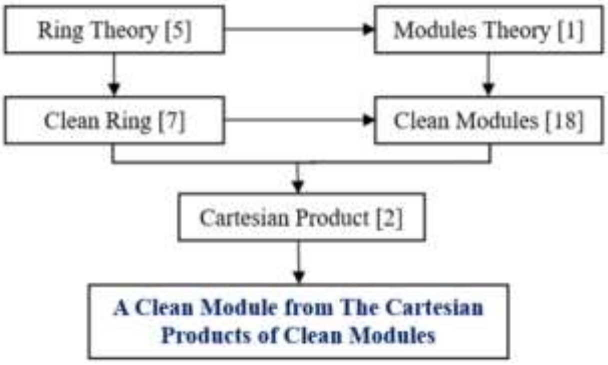

Figure 1. The state of the art of this research

III. RESULT AND DISCUSSION

This section presents the results of research on clean modules constructed through the external direct sum of clean modules. The concept of isomorphism shows that there are similarities in properties between two different structures. A different algebraic structure has similar properties that depend on the relationship between them. Based on this fact, it can be shown that a clean module can be formed from the external direct sum of existing clean modules. The external direct sum of a module is an ordered pair of two different modules that form a new module. In this case, if we have M_1, M_2 modules over the ring R, then $M_1 \times M_2$ is also a module over the ring R, which is also called the external direct sum. More deeply, the R-module structure $M_1 \times M_2$ can form an $End_R(M_1 \times M_2)$. In [1], it is stated that $End_R(M)$ can form a ring structure with addition and function composition operations, by applying the same proof technique used for $End_R(M_1 \times M_2)$, then we can show that $End_R(M_1 \times M_2)$ can also form a ring. On the other hand, an ordered pair can be formed between modules, i.e. R-module $M_1 \times R$-module M_2, we also have $End_R(M_1) \times End_R(M_2)$. According to [1], each of $End_R(M_1)$ and $End_R(M_2)$ are both rings, the Cartesian product of two rings forms a new ring, namely the ring $End_R(M_1) \times End_R(M_2)$. That is, two rings are obtained, namely the ring $End_R(M_1 \times M_2)$ and the ring $End_R(M_1) \times End_R(M_2)$. There is a relationship between the two rings which is presented in the following theorem.

Theorem 1. Let M_1, M_2 be an R-modules. The external direct sum of the modules, R-module $(M_1 \times M_2)$ is a clean module if and only if M_1 and M_2 are clean modules.

Proof.

($\Longrightarrow$) It is known that $(M_1 \times M_2)$ is a clean module, its mean that ring $End_R(M_1 \times M_2)$ is a clean. We prove that M_1, M_2 are clean modules by showing that $End_R(M_1 \times M_2) \cong End_R(M_1) \times End_R(M_2)$. A mapping is defined as follows:

$$\alpha: End_R(M_1 \times M_2) \to End_R(M_1) \times End_R(M_2)$$

$$\alpha(h,k) \mapsto h \times k$$

with $h \times k \in End_R(M_1) \times End_R(M_2)$, for any $(m_1, m_2) \in M_1 \times M_2$, then

$$(h \times k)(m_1, m_2) = \big(h(m_1), k(m_2)\big)$$

It will be proven that the mapping α above is a ring isomorphism.

a. The mapping α is preserves the addition operation.
   It is obtained that for every $(m_1, m_2) \in M_1 \times M_2$ we get $(h_1 + h_2) \times (k_1 + k_2) = (h_1 \times k_1) + (h_2 \times k_2)$ or $\big((h_1, k_1) + (h_2, k_2)\big) = \alpha(h_1, k_1) + \alpha(h_2, k_2)$. This means that the mapping α preserves the addition operation.
b. The mapping α is preserves the function composition operation.
   It is obtained that for every $(m_1, m_2) \in M_1 \times M_2$ we get $(h_1 \circ h_2) \times (k_1 \circ k_2) = (h_1 \times k_1) \circ (h_2 \times k_2)$ or $\alpha\big((h_1, k_1) \circ (h_2, k_2)\big) = \alpha(h_1, k_1) \circ \alpha(h_2, k_2)$. So, this is proven that the mapping α is preserves the function composition operation.
c. The mapping α is injective.
   Take any $(h_1, k_1), (h_2, k_2) \in End_R(M_1 \times M_2)$, if $\alpha(h_1, k_1) = \alpha(h_2, k_2)$, then $h_1 \times k_1 = h_2 \times k_2$. We will investigate that $(h_1, k_1) = (h_2, k_2)$, assuming that $h_1 = h_2$ and $k_1 = k_2$, for each $(m_1, m_2) \in M_1 \times M_2$, then holds
   $$\begin{aligned}(h_1 \times k_1)(m_1, m_2) &= \big(h_1(m_1), k_1(m_2)\big)\\ &= \big(h_2(m_1), k_2(m_2)\big)\\ &= (h_2 \times k_2)(m_1, m_2)\end{aligned}$$
   Based on the similarity of two functions, for each $(m_1, m_2) \in M_1 \times M_2$, it is obtained that $(h_1 \times k_1) = (h_2 \times k_2)$, meaning that based on the definition of the mapping α, we get $(h_1, k_1) = (h_2, k_2)$. So, it is proven that the mapping α is injective.
d. The mapping α is surjective.
   We shown that for $h \times k \in End_R(M_1) \times End_R(M_2)$, there exists $(h, k) \in End_R(M_1 \times M_2)$, then
   $$\alpha(h,k) = h \times k$$
   For every $(m_1, m_2) \in M_1 \times M_2$, it holds
   $$\begin{aligned}(h \times k)(m_1, m_2) &= \big(h(m_1), k(m_2)\big)\\ &= (h,k)(m_1, m_2)\end{aligned}$$
   Thus, for every $h \times k \in End_R(M_1) \times End_R(M_2)$, there exists $(h, k) \in End_R(M_1 \times M_2)$. It holds that a mapping α is surjective.

Using the facts on a, b, c, and d, it is satisfied that the mapping α is a ring isomorphism, meaning $End_R(M_1 \times M_2) \cong End_R(M_1) \times End_R(M_2)$.

Based on the fact obtained that $End_R(M_1 \times M_2) \cong End_R(M_1) \times End_R(M_2)$ and since it is known that $End_R(M_1 \times M_2)$ is a clean module, then $End_R(M_1) \times End_R(M_2)$ is also a clean ring. Furthermore, based on [18] which states that the Cartesian product of a ring is a clean ring if and only if each of its constituent rings is a clean ring,

($\Longleftarrow$) It is known that M_1 and M_2 are clean modules. It will be investigated that the R-module $(M_1 \times M_2)$ is a clean module by showing that $End_R(M_1 \times M_2)$ is a clean ring. Based on the known, it is obtained that $End_R(M_1)$ and $End_R(M_2)$ are clean rings, and based on [18] it is obtained that $End_R(M_1) \times End_R(M_2)$ is also a clean ring. As a result of $End_R(M_1 \times M_2) \cong End_R(M_1) \times End_R(M_2)$, because there is a clean ring $End_R(M_1) \times End_R(M_2)$, then the ring $End_R(M_1 \times M_2)$ is a clean ring. So, R-module $(M_1 \times M_2)$ is a clean modules.

Theorem 2. Let $M_1, M_2, \ldots, M_n$ be an R-module, for $n \in \mathbb{N}$. The external direct sum of the modules, the R-modules $(M_1 \times M_2 \times \ldots \times M_n)$ is a clean modules if and only if M_1, $M_2, \ldots, M_n$ are each a clean modules.

Proof.

($\Longrightarrow$) It is known that the R-module $(M_1 \times M_2 \times \ldots \times M_n)$ is clean, meaning that $End_R(M_1 \times M_2 \times \ldots \times M_n)$ is a clean ring. It is proved that $M_1, M_2, \ldots, M_n$ are each clean R-modules by proving that $End_R(M_1) \times \ldots \times End_R(M_n) \cong End_R(M_1 \times \ldots \times M_n)$. Referring to Theorem 1, it shows that $End_R(M_1) \times End_R(M_2) \cong End_R(M_1 \times M_2)$. Next, it is assumed that

$$End_R(M_1) \times \ldots \times End_R(M_k) \cong End_R(M_1 \times \ldots \times M_k)$$

for every $n = k \geq 1 \in \mathbb{N}$, then it is proven that

$$End_R(M_1) \times \ldots \times End_R(M_k) \times End_R(M_{k+1}) \cong End_R(M_1 \times \ldots \times M_k \times M_{k+1}).$$

Since $End_R(M_1) \times \ldots \times End_R(M_k) \cong End_R(M_1 \times \ldots \times M_k)$, then

$$End_R(M_1) \times \ldots \times End_R(M_k) \times End_R(M_{k+1}) = End_R(M_1 \times \ldots \times M_k) \times End_R(M_{k+1})$$

Furthermore, it's shown that $End_R(M_1 \times \ldots \times M_k) \times End_R(M_{k+1}) \cong End_R(M_1 \times \ldots \times M_k \times M_{k+1})$. Given any $(h_1, \ldots, h_k, h_{k+1}) \in \mathrm{End_R}(\mathrm{M_1} \times \ldots \times \mathrm{M_k} \times \mathrm{M_{k+1}})$, A mapping is defined as follows:

$$\beta : (\mathrm{M_1} \times \ldots \times \mathrm{M_k} \times \mathrm{M_{k+1}}) \to \mathrm{End_R}(\mathrm{M_1} \times \ldots \times \mathrm{M_k}) \times \mathrm{End_R}(\mathrm{M_{k+1}})$$

$$\beta(\mathrm{h_1}, \ldots, \mathrm{h_k}, \mathrm{h_{k+1}}) \longmapsto (\mathrm{h_1}, \ldots, \mathrm{h_k}) \times \mathrm{h_{k+1}}$$

for every $(\mathrm{m_1}, \ldots, \mathrm{m_k}, \mathrm{m_{k+1}}) \in \mathrm{M_1} \times \ldots \times \mathrm{M_k} \times \mathrm{M_{k+1}}$, then

$$\big((\mathrm{h_1}, \ldots, \mathrm{h_k}) \times \mathrm{h_{k+1}}\big)(\mathrm{m_1}, \ldots, \mathrm{m_k}, \mathrm{m_{k+1}}) = \big((\mathrm{h_1}, \ldots, \mathrm{h_k})(\mathrm{m_1}, \ldots, \mathrm{m_k}), (\mathrm{h_{k+1}})(\mathrm{m_{k+1}})\big)$$

Using the same analogy in Theorem 1, it is obtained that $End_R(M_1) \times \ldots \times End_R(M_k) \times End_R(M_{k+1}) = End_R(M_1 \times \ldots \times M_k) \times End_R(M_{k+1})$ That is, it is proven that $End_R(M_1) \times End_R(M_2) \times \ldots \times End_R(M_n) \cong End_R(M_1 \times M_2 \times \ldots \times M_n)$ for $n \in \mathbb{N}$.

Based on the facts, obtained $End_R(M_1) \times End_R(M_2) \times \ldots \times End_R(M_n) \cong End_R(M_1 \times M_2 \times \ldots \times M_n)$, for $n \in \mathbb{N}$ and it is known that $End_R(M_1 \times M_2 \times \ldots \times M_n)$ is a clean module, then $End_R(M_1) \times End_R(M_2) \times \ldots \times End_R(M_n)$ is also a clean ring. Furthermore, based on [18] which states that the Cartesian product of a ring is a clean ring if and only if each of its constituent rings is a clean ring, since the ring $End_R(M_1) \times End_R(M_2) \times \ldots \times End_R(M_n)$ is a clean ring, then the rings $End_R(M_1)$, $End_R(M_2)$, $\ldots$, $End_R(M_n)$ are clean rings. Consequently, $M_1, M_2, \ldots, M_n$ are each a clean module.

($\Longleftarrow$) It is known that $M_1, M_2, \ldots, M_n$ are each clean modules. We investigate that the R-module $(M_1 \times M_2 \times \ldots \times M_n)$ is a clean module by showing that $End_R(M_1 \times M_2 \times \ldots \times M_n)$ is a clean ring. By the same analogy as in Theorem 1, we prove that the R-module $(M_1 \times M_2 \times \ldots \times M_n)$ is a clean modules.

By adding one property to the net element, where the building elements of the net element, namely the unit and idempotent elements of $End_R(M)$ have the commutative property in the composition operation of their functions, then the net element can be called strongly clean. This means that a strongly clean module can also be constructed from the external direct sum of a strongly clean module, which is stated in the following theorem.

Theorem 3. Let Let $M_1, M_2, \ldots, M_n$ be an R-module, for $n \in \mathbb{N}$. The external direct sum of the modules, the R-modules $(M_1 \times M_2 \times \ldots \times M_n)$ is a strongly clean modules if and only if $M_1, M_2, \ldots, M_n$ are each strongly clean modules.

Proof.

This is analogous to the previous proof..

IV. CONCLUTION

In this study, it is found that if there is an R-module $M_1 \times M_2$, then $End_R(M_1 \times M_2)$ is a ring, which can be generalized into a ring $End_R(M_1 \times M_2 \times \ldots \times M_n)$ for any $n \in \mathbb{N}$. In addition, it is found that the ring $End_R(M_1 \times M_2)$ is isomorphic to the ring $End_R(M_1) \times End_R(M_2)$. Consequently, if M_1 and M_2 are both clean modules, then the external direct sum between them is also a clean module. By adding one commutative property to the function composition operation of unit elements and idempotents element, the external direct sum module is strongly clean. Suggestions for further research, can be investigated the net properties of the internal direct sum of the modules.

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