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Analysis of Conctruction Cayley's Theorem on Groups, Semigroups Dan Monoids

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ABSTRACT

Cayley's theorem discusses the embedding operations between algebraic structures which are carried out by showing that there is an injective homomorphism between these algebraic structures. Each of algebraic structure has different character, so the aim of this article is to discuss how to construct Cayley's Theorem on Groups, semigroups and monoids.

Construction of Cayley's Theorem on group G begins by constructing group TG from set of functions from G to G. The next step is to show that there is an injective homomorphism from G to TG. Construction of Cayley's Theorem on semigroups and monoids is not the same as Cayley's Theorem on groups. There are several conditions that must be added.

KEYWORDS: Cayley, Grup, Semigroup, Monoid, Homomorphism.

I. INTRODUCTION

Let $(U,*)$ and $(V,\blacksquare)$ be two algebraic structures with corresponding binary operations. A mapping $\phi : U \rightarrow V$ is called a homomorphism [1, 2,4,6, 9,11] , if

$$\phi(u_1 * u_2) = \phi(u_1) \blacksquare \phi(u_2)$$

for all $u_1, u_2 \in U$

Hence, a homomorphism respects the product of one algebraic structure while transferring elements to some other structure.

A homomorphisma $\phi : U \rightarrow V$ is called an embedding if $\phi(u_1) = \phi(u_2)$ implies $u_1 = u_2$ for all $u_1, u_2 \in U$. The statement is equivalent to for all $u_1, u_2 \in U$, if $u_1 \neq u_2$ implies

$$\phi(u_1) \neq \phi(u_2)$$

A semigroup is an algebraic structure with one associative binary operation [3,5,10,12,17] monoid is a semigroup that meets the conditions for the existence of an identity element [2,7,8,15,16], while a group is a monoid that meets the conditions for the existence of an inverse or reverse element.

Cayley's Theorem, a foundational result in group theory, states that every group is isomorphic to a subgroup of a permutation group, also known as a symmetric group.[13]

Cayley's Theorem, in the context of semigroups, This theorem is an extension of Cayley's Theorem for groups, which states that every group can be embedded into a permutation group. [14]

Cayley's theorem for monoids states that every monoid can be embedded in the tranformation monoid of all self-maps on a set.

Construction of insertion operation presented in Cayley's Theorem for each group, semigroup and monoid algebraic structure has several differences. Therefore this paper explains the construction analysis.

II. METHOD AND DISSCUSION

In general, the method used for algebraic structures of groups, semigroups and monoids is as follows:

1. Building a set of functions on the algebraic structure X
2. Add operations at the set of functions obtained for obtain the algebraic structure Y, which has the same structure as X
3. Constructing an injective homomorphism from X to Y by using embedding method

This paper explains the motivation for how Cayley's Theorem was constructed, which has never been discussed before. In many papers, the Cayley's Theorem at algebraic structure of groups, semigroups and monoid is directly given.

If given a group $(G,*)$ then a set of functions from G to G can be constructed which is denoted by T_G. It is hoped that T_G which is equipped with the "$\circ$" function composition operation will form a group, so that a homomorphism can be built from G to T_G. However, the existence of the inverse of every function in T_G cannot be guaranteed, so $(T_G, \circ)$ is not a group. Furthermore, the set of functions in T_G is only limited for bijective functions. The set of bijective functions from G to G is denoted by T_{GBij}. Function "$\circ$" in T_{GBij} is

$$\begin{aligned} \circ : T_{GBij} \times T_{GBij} &\rightarrow T_{GBij} \\ (g,h) &\rightarrow g \circ h \end{aligned}$$

For all $g, h \in T_{GBij}$. It can be shown that $(T_{GBij}, \circ)$ is a group. Up to this point, groups $(G,*)$ and $(T_{GBij}, \circ)$ have been constructed. Next, the function

$\varphi : (G,*) \rightarrow (T_{GBij}, \circ)$ is built with the following definition:

$$\varphi_g(h) = g \circ h$$

for all $g, h \in G$

Remembering that $T_{GBij} \subseteq T_G$, so it must first be confirmed that if $g \in G$ then $\varphi_g \in T_{GBij}$. In other words, if $g \in G$ then φ_g is a bijective function from G to G.

First shown that if $g \in G$ then φ_g is an injective function. Proved by if $h_1, h_2 \in G$ with $\varphi_g(h_1) = \varphi_g(h_2)$ then $h_1 = h_2$.

$$\varphi_g(h_1) = g \circ h_1$$

and

$$\varphi_g(h_2) = g \circ h_2$$

Because of $\varphi_g(h_1) = \varphi_g(h_2)$,
so $g \circ h_1 = g \circ h_2$.
Let $g \in G$ and G grup, then $g^{-1} \circ g \circ h_1 = g^{-1} \circ g \circ h_2$
So $h_1 = h_2$

Next it is shown that if $g \in G$ then φ_g is a surjective function. In other words, if $h \in G$ then it must be shown that there exists $h' \in G$ such that $\varphi_g(h') = h$

For $h \in G$, there exists $h' = g^{-1} \circ h$ such that
$\varphi_g(h') = g \circ h' = g \circ (g^{-1} \circ h) = (g \circ g^{-1}) \circ h = h$
So it is certain that if $g \in G$ then $\varphi_g \in T_{GBij}$

Next, it is proven that $\varphi : (G,*) \to (T_{GBij}, \circ)$ is homomorphism, namely by proving that for all $g_1, g_2 \in G$ holds

$$\varphi_{g_1 * g_2} = \varphi_{g_1} \circ \varphi_{g_2}$$

The proof is carried out by applying it to an $h \in G$ as follows

$$\varphi_{g_1 * g_2} = \varphi_{g_1 * g_2}(h) = (g_1 * g_2) \circ h = g_1 * (g_2 \circ h) = (\varphi_{g_1} \circ \varphi_{g_2})(h) = \varphi_{g_1} \circ \varphi_{g_2}$$

Finally it must be shown that $\varphi : (G,*) \to (T_{GBij}, \circ)$ is injective. In other words, if $g_1, g_2 \in G$ and $\varphi_{g_1} = \varphi_{g_2}$ then $g_1 = g_2$.

Theorem 4.1.
If G is a group, then we can find the injective homomorphism $\varphi : (G,*) \to (T_{GBij}, \circ)$ with the following definition:

$$\varphi_g(h) = g \circ h$$

for all $g, h \in G$

Next, we discuss how to construct Cayley's Theorem in semigroup algebraic structures. Given the semigroup $(S,*)$ then we can construct a set of functions ρ from S to S which is denoted by T_S. Because the composition operation " $\circ$ " on T_S is associative, the set $(T_S, \circ)$ forms a semigroup, so it is hoped that an injective homomorphism can be built from S to T_S. The function $\rho : S \to S$ is defined by

$$\rho_s(r) = s * r$$

for all $s, r \in S$. Next, the function $\varphi : (S,*) \to (T_S, \circ)$ is built with the following definition:

$$\varphi_s = \rho_s$$

for all $s \in S$. To show that S can be inserted into T_S it must be shown that φ is an injective homomorphism

Investigate whether φ is injective. It is known that $\varphi_s = \varphi_t$ for all $s, t \in S$. Using the definition of the φ function, we obtain $\rho_s = \rho_t$ for all $s, t \in S$. This means

$$\rho_s(x) = \rho_t(x)$$

for all $s, t, x \in S$. For $x = 1$ we can get $\rho_s(1) = \rho_t(1)$, namely $s * 1 = t * 1$. So that $s = t$

However, keep in mind that semigroup S does not have a unit element. Therefore it is necessary to involve the set S' as follows:

$$S' = \begin{cases} S, \mathit{if}\ S\ \mathit{monoid} \\ \quad S \cup \{1\} \end{cases}$$

with 1 *is not in* S *but has properties as a unit element in* S

So we can construct the function $\vartheta : S' \to S'$ with ⟦$\vartheta_s(r) = s * r$ for all $s, r \in S'$. Likewise, the function $\omega : (S,*) \to (T_{S'}, \circ)$ with the following definition:

$$\omega_s = \vartheta_s$$

for all s∈S.

Finally, it is proven that ω is homomorphism, namely for all $s, t \in S$ satisfies

$$\omega(s * t) = \omega(s) \circ \omega(s).$$

However, previously taken $\vartheta_s, \vartheta_t \in T_{S'}$. Using associative property of the operation " $*$ " and definition of function ϑ, then for all $x \in S'$ we hold

$$\begin{aligned} (\vartheta_s \circ \vartheta_t)(x) &= \vartheta_s(\vartheta_t(x)) \\ &= \vartheta_s(t * x) \\ &= s * (t * x) \\ = (s * t) * x &= \vartheta_{s*t}(x) \text{........(1)} \end{aligned}$$

Next, we prove ω is homomorphism. Let $s, t \in S$. According to definition of function ω then

$$\omega_{s*t} = \vartheta_{s*t}$$

Using equation (1) and definition of ω, we obtain that for every $s, t \in S$

$$\begin{aligned} \omega_{s*t} &= \vartheta_s \circ \vartheta_t \\ &= \omega_s \circ \omega_t \end{aligned}$$

The above description is a stages in constructing Cayley's Theorem on a semigroup algebraic structure. So Cayley's Theorem on semigroups can be stated as follows:

Theorem 4.2.
If S is a semigroup and

$$S' = \begin{cases} S, \mathit{if}\ S\ \mathit{monoid} \\ \quad S \cup \{1\}, \end{cases}$$

with 1 is not in S but has properties as a unit element in S, then we can find the injective homomorphism $\omega : (S,*) \to (T_{S'}, \circ)$ with definition

$$\omega_s = \vartheta_s$$

for all $s \in S$ and $\vartheta_s \in T_{S'}$.

The final question is how to construct Cayley's Theorem on monoid algebraic structures.

It is known that the monoid $(M,*)$. The same when constructing Cayley's Theorem on groups and semigroups, the first step is to construct the function $\mu : M \to M$ with the definition

$$\mu_m(n) = m * n$$

for all $m, n \in M$

Set of all μ functions is denoted by T_M. Algebraic structure $(T_M, \circ)$ is a monoid. Apart from that, the composition operation " $\circ$ " on T_M is associative and for all $1 \in M$ there is $i_1 : M \to M$ by definition

$$i_1(r) = 1 * r = r$$

for all $r \in M$.

If $m, n \in M$, then for all $\mu_m \in T_M$ one can find $i_1 \in T_M$ such that

$$(\mu_m \circ i_1) = (\mu_m \circ i_1)(r) = \mu_m(i_1(r)) = \mu_m(r) = \mu_m.$$

By following the steps to construct Cayley's Theorem on groups and semigroups, we then prove that $(M,*)$ can be inserted into $(T_M,\circ)$ by showing that there is an injective homomorphism $\theta : (M,*) \to (T_M,\circ)$ by definition

$$\theta_m = \mu_m$$

for all $m \in M$.

It has been discussed when constructing Cayley's Theorem on semigroups, that there is an injective homomorphism $\omega : (S,*) \to (T_{S'},\circ)$ with the following definition:

$$\omega_s = \vartheta_s$$

for all $s \in S$.

Theorem 4.3.

If M is monoid, then an insertion from M to T_M can be found. Where T_M is a set of functions from M to M.

III. CONCLUTION

The set of functions constructed when constructing Cayley's Theorem on groups is restricted to only bijective functions. When the function composition operation is added, it can form an group. Meanwhile, construction of Cayley's Theorem for semigroups is carried out by combining the element "1" which has the same properties as the unit element. This is done so that the injective properties of the homomorphism from the semigroup to the set of functions in the semigroup are fulfilled. Finally, the construction of Cayley's theorem on monids is simply proven by showing that the homomorphism map of the unit element is an identity function.

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